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Four-Plate Pick-up Capacitance and Sensitivity Calculations

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Abstract

The goal of calculations presented in this note is to find and maximize the sensitivity of a quadrupole pick-up currently being designed. The calculations are made using the CERN package Poisson[1]. The range of electrode widths under consideration is:$.006m < w < .03$. Studies indicate that sensitivity is maximized in this range by the smallest width plate if the electronics contributes negligible capacitance. The plate size for which the sensitivity is optimized increases with increasing electronics capacitance. As well, $6cm$ and $8cm$ outer shells of both circular and square geometry are considered. An $8cm$ square quadrupole pickup yields the higher sensitivity.

Introduction

Assuming the absence of free charge, Poisson's equation:

$$\nabla^2 V(x,y) = -\rho/\epsilon$$

reduces to Laplace's equation:

$$\nabla^2 V(x,y) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$$

The general solution to this equation is

$$V(x,y) = V_0 + V_1 x + V_2 y + V_3 xy + V_4(x^2 - y^2)$$

For the geometry of the pick-up under consideration(see figure 1), $V_3 = 0$ by symmetry. As well, any configuration of potentials can be written as a linear combination of the configurations of figure 2 with each corresponding to one of the terms in $V(x,y)$.

Induced Charge

Green's reciprocity theorem[2] is as follows: If total charges $q_1, q_2, ..., q_n$ on n conductors produce potentials $V_1, V_2, ..., V_n$ and if charges $q_1', q_2', ..., q_n'$ on the same set of n conductors produce potentials $V_1', V_2', ..., V_n'$, then

$$\sum_{i=1}^{n} q_i V_i' = \sum_{i=1}^{n} q_i' V_i$$

Using this theorem, it is possible to extract the induced charge on a plate by using the configurations of figure 3. In figure 3a, all plates are grounded and there is a charge q at location (x, y) with voltage V(x,y). In figure 3b, plate 1 has 1 volt and $q'(x, y) = 0$. Plugging into Green's reciprocity theorem yields: $qV'(x, y) + q_1 = 0$. Hence the charge q_1 induced by q in figure 3a is $-V'(xy)q$. Applying Green's theorem to figures 3a and figure 2a yields the induced charge for the quadrupole term: $-qV'(x, y) = q_1 - q_2 + q_3 - q_4$. For the sum mode, figures 3a and 2b yield $-qV'(x, y) = q_1 + q_2 + q_3 + q_4$. Figure 3a with 2c and 2d yield the dipole terms' induced charges which are respectively $-qV'(x, y) = q_1 - q_3$ and $-qV'(x, y) = q_2 - q_4$. Hence, in each mode, $-V(x, y)$ is the total induced charge fraction. The Poisson package readily provides V(x,y) for a given geometry and will be used in the sensitivity calculations.

Capacitance

Inter-plate capacitance and capacitance between a particular electrode and the exterior both affect the magnitude of the electrode signal. A general matrix method exists for calculating the overall capacitance of the system.

The capacitance matrix consists of diagonal terms C_{ii} known as capacities and off-diagonals C_{ij} known as coefficients of inductance. From $Q_i = \sum_{j=1}^{n} C_{ij} V_j$

and symmetry considerations, C_{ij} will be extracted. For the configuration:

$$V_1 = 1, \; V_2 = V_3 = V_4 = 0$$

we have

$$Q_1 = C_{11}, \; Q_2 = C_{21}, \; Q_3 = C_{31} \; and \; Q_4 = C_{41}$$

By symmetry,

$$C_{11} = C_{22} = C_{33} = C_{44}$$

$$C_{14} = C_{12} = C_{21} = C_{23} = C_{32} = C_{34} = C_{43} = C_{41}$$

and

$$C_{13} = C_{31} = C_{24} = C_{42}$$

Note that Gauss' law expressed in integral form is $\int_S \vec{E} \cdot \hat{n} = Q/\epsilon_0$. Gaussian integrals around the plates yield the Q_i.

In figure 4, the effective capacitance for the sum,dipole and quadrupole modes is given for 6cm and 8cm outer shell widths as a function of electrode plate width. Note that both square and circular geometries are considered(see figure 1). Capacitance is smaller for larger outer shell size. As well, capacitance is smaller for smaller electrode widths.

Sensitivity

Given the induced plate charges Q_i and capacitance matrix C_{ij}, one may obtain the V_j by inverting C_{ij}. Recalling that by reciprocity, the induced charge ratio $q_{induced}/q_{beam}$ is -V(x,y) where

$$V(x, y) = V_0 + V_1 x + V_2 y + V_3 xy + V_4(x^2 - y^2)$$

it is natural to define the sensitivity of a quadrupole pickup as

$$S_{sum\ mode} = \frac{V_0}{C/l}\ (m/pC)$$

$$S_{dipole\ mode} = \frac{(V_1\ or\ V_2)}{C/l}\ (1/pC)$$

$$S_{quadrupole\ mode} = \frac{V_4}{C/l}\ (1/m - pC)$$

To understand the units, recall that $V(x,y)$ is the potential normalized to 1 volt. The values calculated for this study may be found in tables 1 and 2. The explicit calculation of V_{out} is left to the appendix. Using typical parameter values, a quadrupole mode output voltage of $5.7 \times 10^{-5}V$ is obtained.

Optimizing Geometry with Electronic Capacitance

In figure 5, the output voltage vs. plate width is plotted assuming a plate length of $.5m$ and using the parameter values from the appendix. A range of electronic capacitance values is considered in the figure. The effective capacitance per unit length is

$$C_{eff} = (C_{electronics}/l_{detector} + C_{geometry/unit\ length})$$

From the appendix,

$$V_{quadrupole} = \frac{V_4}{4C/l}\frac{I_{beam}}{c}(a^2 - b^2)$$

where a and b are the beam's elliptical cross-section semi-minor and major axis dimensions and we replace C/l with C_{eff}.

One can see from the figure that the sensitivity peaks at a larger plate width when the electronic capacitance has a larger value.

Conclusions

The geometrical capacitance of the system is minimized for larger outer shells and smaller electrode widths. When one introduces electronics capacitance, however, the maximum sensitivity is obtained for larger plate widths since this results in larger geometrical capacitance values.

References

1. R. Holsinger and C. Iselin,"The CERN-Poisson Program Package(Poiscr-T604)",CERN Computer Centre Program Library,1984.

2. V.V. Batygin and I.N. Toptygin,"Problems in Electrostatics", Academic Press Inc.(London)Ltd.(1964)

Appendix

By reciprocity, the induced charge ratio $q_{induced}/q_{beam}$ is -V(x,y). V(x,y) has the form

$$V(x,y) = V_0 + V_1 x + V_2 y + V_3 xy + V_4(x^2 - y^2)$$

To calculate the quadrupole mode output voltage one must use the following expression:

$$V_{out} = \frac{V_4}{C/l} \int \int dxdy \frac{J_{beam}(x,y)}{c}(x^2 - y^2)$$

For the case of an elliptically shaped beam with constant current density whose semi-major and minor axis lengths are b and a respectively this becomes

$$V_{out} = \frac{V_4}{C/l} \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \frac{J_{beam}}{c}(x^2 - y^2)$$

$$= 2b\frac{V_4}{C/l} \int_{-a}^{a} x^2 dx \sqrt{1 - \frac{x^2}{a^2}} \frac{J_{beam}}{c} - \frac{2}{3} b^3 \frac{V_4}{C/l} \int_{-a}^{a} dx \sqrt{1 - \frac{x^2}{a^2}}^{3} \frac{J_{beam}}{c}$$

$$= 2a^3 b \frac{V_4}{C/l} \int_{-\pi/2}^{\pi/2} d\theta sin^2\theta cos^2\theta \frac{J_{beam}}{c} - \frac{2}{3} ab^3 \frac{V_4}{C/l} \int_{-\pi/2}^{\pi/2} d\theta cos^4\theta \frac{J_{beam}}{c}$$

$$= 2a^3 b \frac{V_4}{C/l} \int_{-\pi/2}^{\pi/2} d\theta cos^2\theta \frac{J_{beam}}{c} - (\frac{2}{3} ab^3 + 2a^3 b) \frac{V_4}{C/l} \int_{-\pi/2}^{\pi/2} d\theta cos^4\theta \frac{J_{beam}}{c}$$

Evaluating with use of an integral table yields

$$= \frac{V_4}{4C/l} \frac{\pi ab \times J_{beam} = I_{beam}}{c} (a^2 - b^2)$$

Assuming $I = 200\mu A$, $a = 1mm$, $b = 2.25mm$, $c = 3 \times 10^8$ along with $S_{quad} = V_4/(C/l) = -84.32(1/pC - m)$ for an $8cm$ sq. shell,$6mm$ plate pick-up gives us

$$V_{out} = -8.432 \times 10^{13} \times \frac{.0002}{3 \times 10^8} \frac{(.001^2 - .00225^2)}{4} = 5.7 \times 10^{-5} V$$

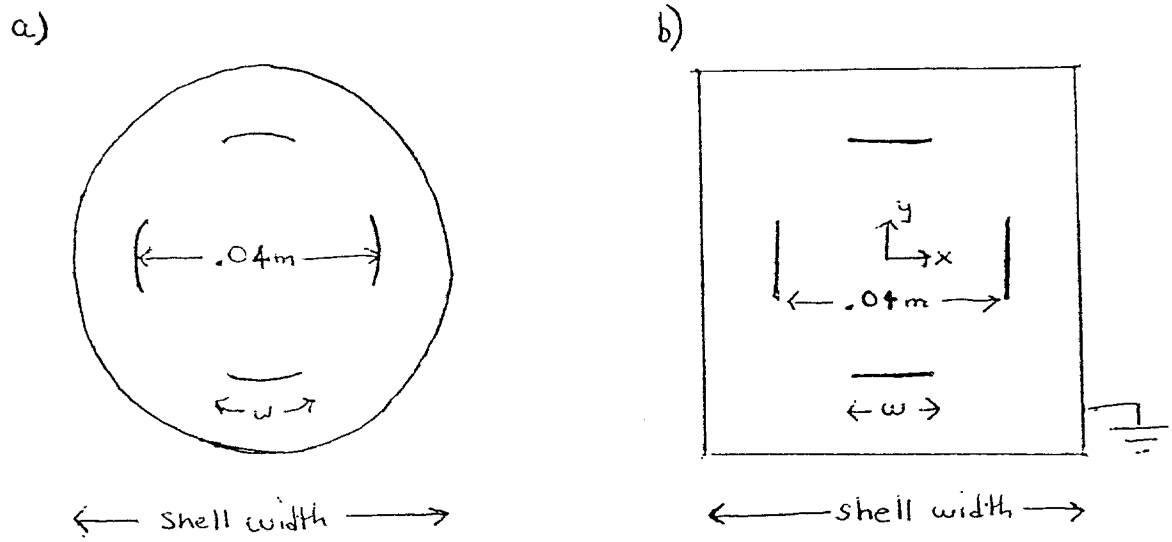

Figure 1. Pick-up geometries under consideration: a)circular, b)square.

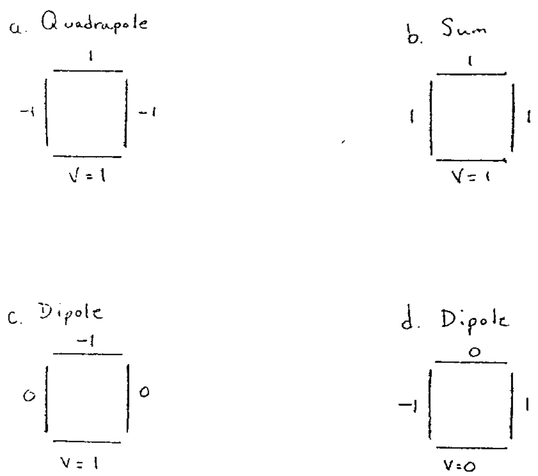

Figure 2. a)quadrapole, b) sum, c)dipole $E_x << E_y$, d)dipole $E_x >> E_y$.

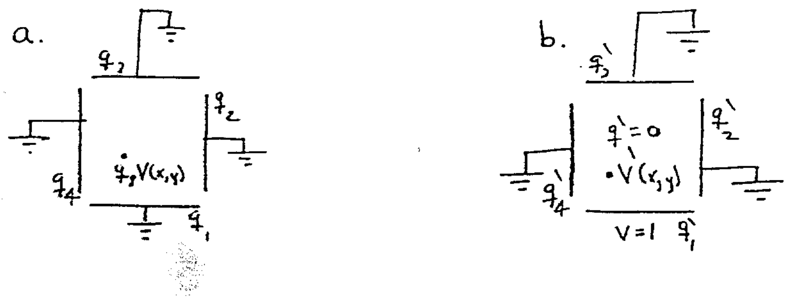

Figure 3. Induced charge configurations.

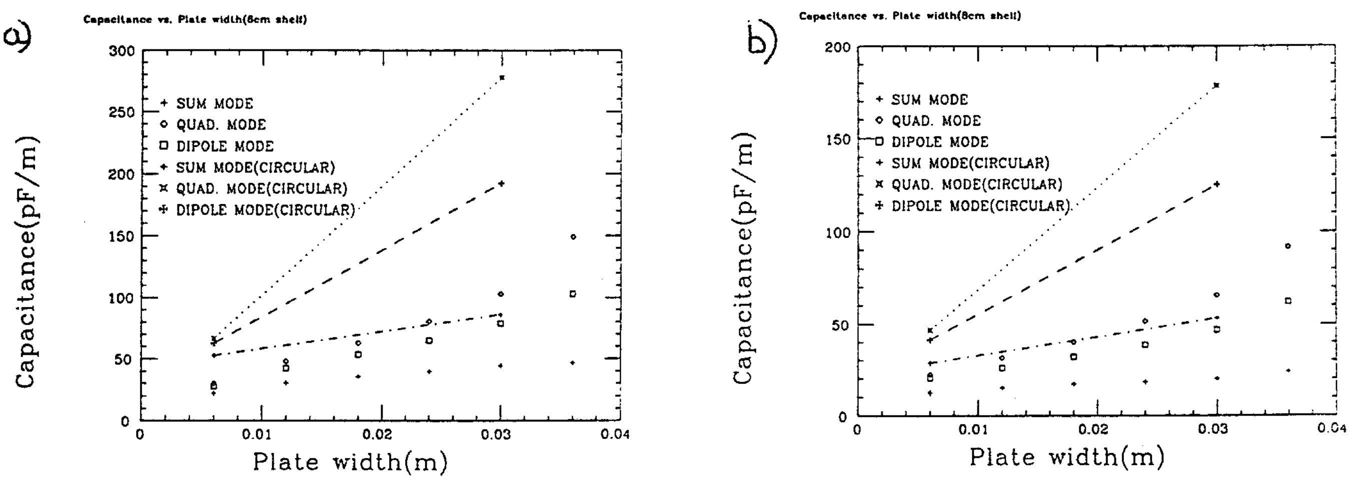

Figure 4. Effective capacitance vs. plate width for a) a 6cm shell, b) an 8cm shell.

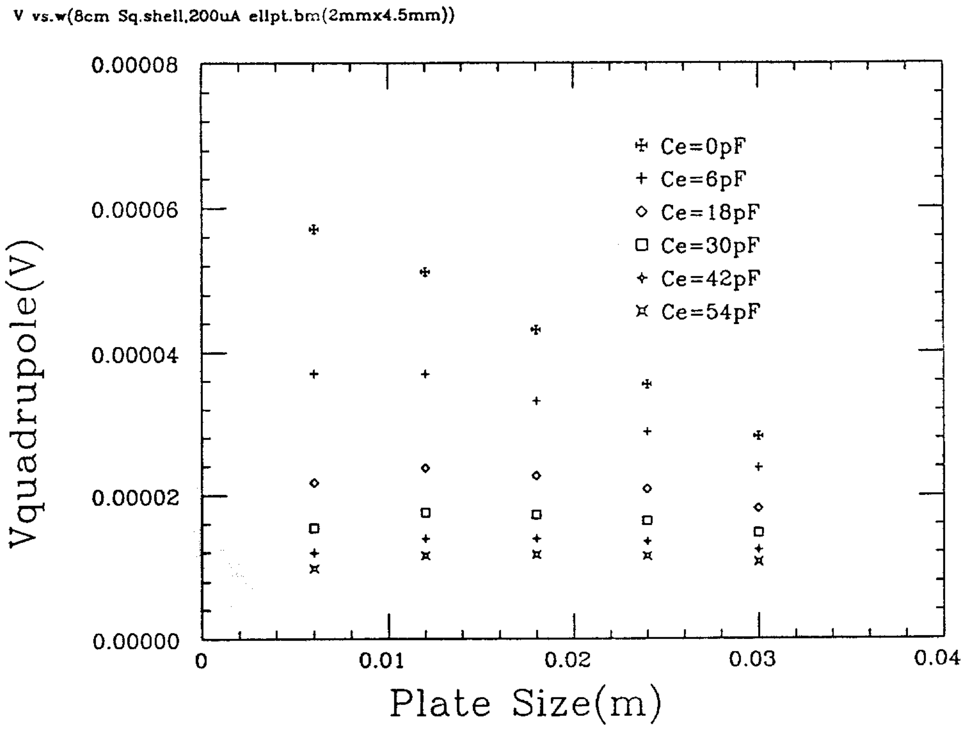

Figure 5. Sensitivity vs. plate width for different electronics capacitance values.

Table 1 Square Geometry Sensitivities

Shell Width(m)	Plate Width(m)	Sum $S(\frac{m}{pC})$	Dipole $S(\frac{1}{pC})$	Quadrupole $S(\frac{1}{pC-m})$
.06	.006	.0331	-.9873	-62.91
.06	.030	.0223	-.5292	-26.67
.08	.006	.0589	-1.348	-84.32
.08	.030	.0499	-.8917	-41.78

Table 2 Circular Geometry Sensitivities

Shell Width(m)	Plate Width(m)	Sum $S(\frac{m}{pC})$	Dipole $S(\frac{1}{pC})$	Quadrupole $S(\frac{1}{pC-m})$
.06	.006	.01056	-.3563	-25.74
.06	.030	.01160	-.2400	-11.19
.08	.006	.02461	-.6469	-40.27
.08	.030	.01880	-.3790	-18.68