Pbar Note #612

Calculation of the Reflection and Transmission Coefficients for Gradual Tapered Waveguide Structures

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Abstract It is required to find the reflection and transmission coefficients for a tapered waveguide. It is assumed that the taper is very long and gradual such that no higher order modes are excited from the discontinuity of the waveguide. Three different methods are presented for finding the reflection coefficient and two methods are presented for finding the transmission coefficient. A traditional differential equation method which finds only the reflection coefficient is presented. An additional set of differential equations are presented which finds both the reflection and transmission coefficients. A FEM (finite element method), which solves for both the reflection and transmission coefficients, is presented with its limitations. An example of a tapered coaxial line and waveguide is presented with the different methods compared and contrasted.

I. Introduction.

It is desired to know the physical behavior for the electromagnetics of a waveguide whose

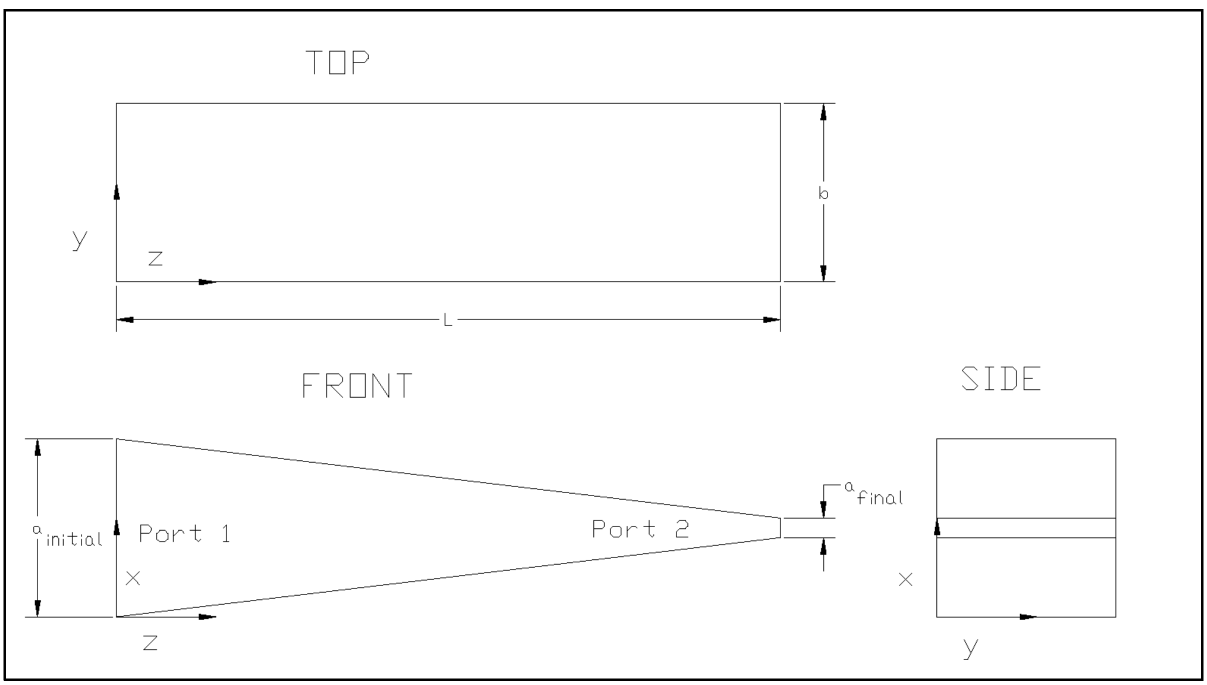

Figure 1. A square waveguide which has a linearly changing cross section as a function of its length. The wave is assumed to enter in Port 1.

cross-section changes as a function of the guide length. Consider the waveguide geometry depicted in Fig. 1. Since the cross-section of the guide undergoes a change, a wave entering the waveguide will undergo multiple reflections and phase changes as the wave progresses along the length of the guide. Since gradual changes do not excite higher-order modes (HOMs), it is not required to do a full electromagnetic analysis of a tapered section. It is only necessary, therefore, to analyze the problem as a single-mode transmission line problem and to not consider mode conversion.

The traditional reflection coefficient problem will be developed in general terms of transmission lines and transmission line parameters and results in a first order differential equation. A general theory for finding both the reflection and transmission coefficients is developed using general transmission line theory and is solved by finding the solutions to two second-order differential equations. It will be shown that only one second order differential equation may be solved if the first differential equation method is used to find the reflection coefficient. A final method is presented using a FEM (finite element method) for calculating both the transmission and reflection coefficients. The problem will then be solved for a coaxial transmission line whose outer diameter changes and also for a waveguide whose dimension changes as depicted in Fig. 1.

II. Analysis

a. Reflection (traditional differential equation method)

Figure 2 is a schematic picture of the device shown in Fig. 1. The parameters in Fig. 2 are as follows: $Z_{initial}$ is known and is the input waveguide wave-impedance, $Z_{in}(z)$ is an unknown function and is the circuit impedance looking into the right-hand portion of the waveguide circuit at position z'=z, $Z(z)$ is a known function and is the tapered-section wave-impedance, Z_{final} is

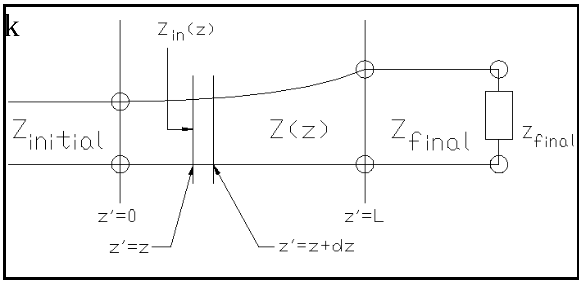

Figure 2. Schematic diagram of the waveguide taper depicted in Fig. 1.

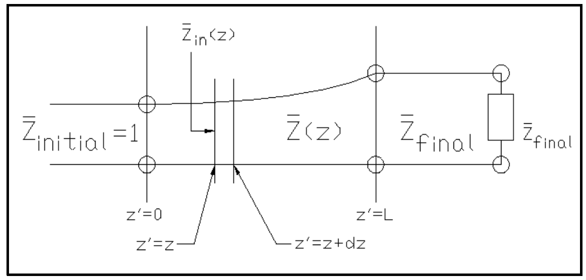

Figure 3. Normalized schematic of Fig. 2. Note that Port 2 is matched.

nown and is the output waveguide wave-impedance, and z_{final} is known and is a circuit impedance which matches to the output waveguide.

An equivalent schematic of Fig. 2 is presented in Fig. 3. The problem of Fig. 3 is simply a normalization to the schematic in Fig. 2. The normalization from Fig. 2 to Fig. 3 are as follows:

$$\bar{Z}_{initial} = \frac{Z_{initial}}{Z_{initial}} = 1, \qquad \bar{Z}_{final} = \frac{Z_{final}}{Z_{initial}}, \qquad \bar{z}_{in}(z) = \frac{z_{in}(z)}{Z_{initial}},$$

$$\bar{Z}(z) = \frac{Z(z)}{Z_{initial}}, \quad \text{and} \quad \bar{z}_{final} = \frac{z_{final}}{Z_{initial}}.$$

An equation for finding $\bar{z}_{in}(z)$ is derived using general transmission line theory [1]. It is assumed that $\bar{z}_{in}(z+dz) = \bar{z}_{in}(z) + d\bar{z}_{in}(z)$, and

$$\bar{z}_{in}(z) = \bar{Z}(z)\frac{\left(\bar{z}_{in}(z) + d\bar{z}_{in}(z)\right) + i\bar{Z}(z)\tan(\beta(z)dz)}{\bar{Z}(z) + i\left(\bar{z}_{in}(z) + d\bar{z}_{in}(z)\right)\tan(\beta(z)dz)},$$

where $\beta(z)$ is the wave-number in the guide.

Since $\beta(z)$ is finite for all finite frequency[1], the tangent function may be replaced by its small argument formulation and therefore

[1] This assumption can be violated for a waveguide mode operated at its cutoff frequency. A small negative complex part can be added to the frequency to circumvent this numerical problem.

$$\bar{z}_{in}(z) \approx \bar{Z}(z)\frac{\left(\bar{z}_{in}(z)+d\bar{z}_{in}(z)\right)+i\bar{Z}(z)\beta(z)dz}{\bar{Z}(z)+i\left(\bar{z}_{in}(z)+d\bar{z}_{in}(z)\right)\beta(z)dz}$$

$$=\left[\left(\bar{z}_{in}(z)+d\bar{z}_{in}(z)\right)+i\bar{Z}(z)\beta(z)dz\right]\frac{\bar{Z}(z)}{\bar{Z}(z)+i\left(\bar{z}_{in}(z)+d\bar{z}_{in}(z)\right)\beta(z)dz}.$$

Using the binomial expansion of $(1+\alpha)^m \approx (1+m\alpha)$ and neglecting the cross multiplication of the differentials results in:

$$\approx\left[\left(\bar{z}_{in}(z)+d\bar{z}_{in}(z)\right)+i\bar{Z}(z)\beta(z)dz\right]\left(1-i\frac{\bar{z}_{in}(z)\beta(z)dz}{\bar{Z}(z)}\right).$$

Expanding this result and again neglecting the cross multiplication of differentials produces

$$\bar{z}_{in}(z) \approx \bar{z}_{in}(z)+d\bar{z}_{in}(z)+i\bar{Z}(z)\beta(z)dz-i\frac{\bar{z}_{in}(z)^2\beta(z)dz}{\bar{Z}(z)}.$$

This equation is simply a first-order differential equation for $\bar{z}_{in}(z)$. Rewriting the differential equation to a conventional form results in

$$\frac{d\bar{z}_{in}(z)}{dz}=i\beta(z)\left(\frac{\bar{z}_{in}(z)^2}{\bar{Z}(z)}-\bar{Z}(z)\right). \tag{1}$$

Another equation from general transmission line theory [2] is

$$\bar{z}_{in}(z)=\bar{Z}(z)\frac{1+s_{11}(z)}{1-s_{11}(z)}. \tag{2}$$

Taking the derivative of this equation yields

$$\frac{d\bar{z}_{in}(z)}{dz}=\frac{2\bar{Z}(z)}{(1-s_{11}(z))^2}\frac{ds_{11}(z)}{dz}+\frac{1+s_{11}(z)}{1-s_{11}(z)}\frac{d\bar{Z}(z)}{dz}. \tag{3}$$

The relationship in Eq. 1 can therefore be equated to the relationship in Eq. 3, which results in

$$\frac{2\bar{Z}(z)}{(1-s_{11}(z))^2}\frac{ds_{11}(z)}{dz}+\frac{1+s_{11}(z)}{1-s_{11}(z)}\frac{d\bar{Z}(z)}{dz}=i\beta(z)\left(\frac{\bar{z}_{in}(z)^2}{\bar{Z}(z)}-\bar{Z}(z)\right). \tag{4}$$

Inserting the relationship of Eq. 2 into Eq. 4 results in

$$\frac{2\overline{Z}(z)}{(1-s_{11}(z))^2}\frac{ds_{11}(z)}{dz}+\frac{1+s_{11}(z)}{1-s_{11}(z)}\frac{d\overline{Z}(z)}{dz}=i\beta(z)\overline{Z}(z)\left(\frac{4s_{11}(z)}{(1-s_{11}(z))^2}\right),$$

and after multiplying both sides of the equation by $(1\text{-}s_{11}(z))$ results in:

$$\frac{2\overline{Z}(z)}{1-s_{11}(z)}\frac{ds_{11}(z)}{dz}+(1+s_{11}(z))\frac{d\overline{Z}(z)}{dz}=i\beta(z)\overline{Z}(z)\left(\frac{4s_{11}(z)}{1-s_{11}(z)}\right).$$

Rewriting this equation produces

$$\frac{ds_{11}(z)}{dz}=-\frac{1-s_{11}(z)^2}{2\overline{Z}(z)}\frac{d\overline{Z}(z)}{dz}+2i\beta(z)s_{11}(z).$$

This equation can be further compressed using the relationship that

$$\frac{1}{f(z)}\frac{df(z)}{dz}=\frac{d\ln(f(z))}{dz},$$

and results in

$$\frac{ds_{11}(z)}{dz}=-\frac{1-s_{11}(z)^2}{2}\frac{d\ln(\overline{Z}(z))}{dz}+2i\beta(z)s_{11}(z). \qquad (5)$$

Equation 5 is an estimate for the reflection coefficient for a tapered waveguide, and is a Ricatti equation. In general, no known general solution exists for Ricatti equations and usually are solved numerically using various computer codes and algorithms. Further information about Ricatti equations may be found, for example, in [3]. Unfortunately, the transmission coefficient cannot be calculated in a similar format because of cancellations due to the form of the input impedance formulation. The results of this equation can be used, however, if the voltage or current waveforms are known as a function of z. A method for finding both the voltage and current waveforms is presented in the next section. Finally, it is important to note that methods exist for designing a particular forcing function (i.e. $\frac{d\ln(\overline{Z})}{dz}$) to produce a desired reflection.

b. Transmission/Reflection (differential equation method)

The transmission and reflection coefficients may be solved for by examining a schematic model of a transmission line. The TEM (transverse electromagnetic) model of a transmission line is depicted in Fig. 4. Both TM (transverse magnetic) and TE (transverse electric) modes require a different circuit model and the results for these modes will be shown in the Example Section III.b. The theory for deriving the TE and TM modes follow the same steps as for the TEM modes and will not be repeated. Using the model of Fig. 4, the following equations are derived governing the voltages and currents:

$$V-(V+\frac{\partial V}{\partial z}dz)=\left(L(z)dz\right)\frac{\partial I}{\partial t}+\left(R(z)dz\right)I$$
$$I-(I+\frac{\partial I}{\partial z}dz)=\left(C(z)dz\right)\frac{\partial V}{\partial t}+\left(G(z)dz\right)V.$$

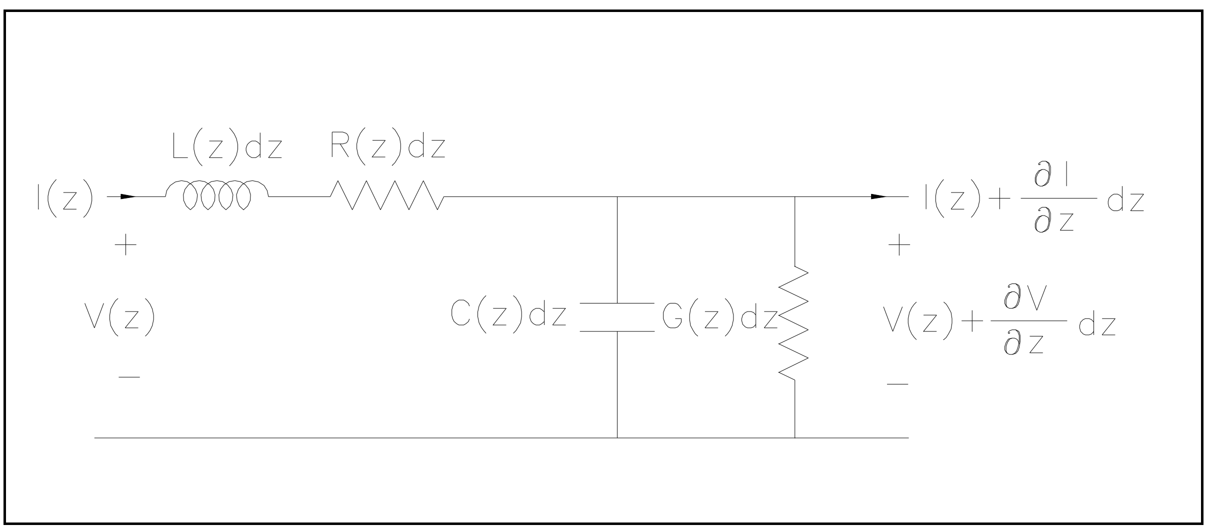

Figure 4. A schematic of a differentially small piece of transmission line. All currents are assumed to be positive in the right-hand direction.

Rewriting these equations and taking the appropriate derivatives of these equations reduces to the previous set of equations into the following set of equations:

$$\frac{\partial V}{\partial z} = -L(z)\frac{\partial I}{\partial t} - R(z)I \tag{6}$$

$$\frac{\partial^2 V}{\partial z \partial t} = -L(z)\frac{\partial^2 I}{\partial t^2} - R(z)\frac{\partial I}{\partial t} \tag{7}$$

$$\frac{\partial I}{\partial z} = -C(z)\frac{\partial V}{\partial t} - G(z)V \tag{8}$$

$$\frac{\partial^2 I}{\partial z \partial t} = -C(z)\frac{\partial^2 V}{\partial t^2} - G(z)\frac{\partial V}{\partial t}. \tag{9}$$

Taking the derivative of Eq. 6 w.r.t. z results in:

$$\frac{\partial^2 V}{\partial z^2} = -\frac{dL(z)}{dz}\frac{\partial I}{\partial t} - L(z)\frac{\partial^2 I}{\partial z \partial t} - R\frac{\partial I}{\partial z} - \frac{dR(z)}{dz}I. \tag{10}$$

Substituting Eqs. 8 and 9 into Eq. 10 results in:

$$\frac{\partial^2 V}{\partial z^2} = -\frac{dL(z)}{dz}\frac{\partial I}{\partial t} - L(z)\left(-C(z)\frac{\partial^2 V}{\partial t^2} - G\frac{\partial V}{\partial t}\right) - R\left(-C(z)\frac{\partial V}{\partial t} - GV\right) - \frac{dR(z)}{dz}I. \tag{11}$$

Taking the Fourier transform[2] of Eq. 11 results in:

$$\frac{\partial^2 \hat{V}}{\partial z^2} = \hat{V}\left(-\omega^2 L(z)C(z) + i\omega\left(L(z)G(z) + R(z)C(z)\right) + R(z)G(z)\right) + \hat{I}\left(-i\omega\frac{dL(z)}{dz} - \frac{dR(z)}{dz}\right).$$

Inserting the Fourier transformed Eq. 6 into the previous equation results in a final equation for the voltage waveform on the transmission line:

$$\frac{\partial^2 \hat{V}}{\partial z^2} - \frac{\partial \hat{V}}{\partial z}\left(\frac{i\omega\frac{dL(z)}{dz} + \frac{dR(z)}{dz}}{i\omega L(z) + R(z)}\right) = \hat{V}\left(-\omega^2 L(z)C(z) + i\omega\left(L(z)G(z) + R(z)C(z)\right) + R(z)G(z)\right). \tag{12}$$

Equation 12 is a second order differential equation which solves for the voltage standing wave

[2]The Fourier transform pair used in this paper is as follows:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \Leftrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

pattern.

The equation for the current is solved for in a similar manner as for the voltage. Taking the derivative of Eq. 8 w.r.t. z results in:

$$\frac{\partial^2 I}{\partial z^2} = -\frac{dC(z)}{dz}\frac{\partial V}{\partial t} - C(z)\frac{\partial^2 V}{\partial z \partial t} - G\frac{\partial V}{\partial z} - \frac{dG(z)}{dz}V. \tag{13}$$

Substituting Eqs. 6 and 7 into Eq. 13 results in:

$$\frac{\partial^2 I}{\partial z^2} = -\frac{dC(z)}{dz}\frac{\partial V}{\partial t} - C(z)\left(-L(z)\frac{\partial^2 I}{\partial t^2} - R\frac{\partial I}{\partial t}\right) - G\left(-L(z)\frac{\partial I}{\partial t} - RI\right) - \frac{dG(z)}{dz}V. \tag{14}$$

Taking the Fourier Transform of Eq. 14 results in:

$$\frac{\partial^2 \hat{I}}{\partial z^2} = \hat{I}\left(-\omega^2 L(z)C(z) + i\omega\left(L(z)G(z) + R(z)C(z)\right) + R(z)G(z)\right) + \hat{V}\left(-i\omega\frac{dC(z)}{dz} - \frac{dG(z)}{dz}\right).$$

Inserting the Fourier Transformed Eq. 8 into the previous equation results in:

$$\frac{\partial^2 \hat{I}}{\partial z^2} - \frac{\partial \hat{I}}{\partial z}\left(\frac{i\omega\frac{dC(z)}{dz} + \frac{dG(z)}{dz}}{i\omega C(z) + G(z)}\right) = \hat{I}\left(-\omega^2 L(z)C(z) + i\omega\left(L(z)G(z) + R(z)C(z)\right) + R(z)G(z)\right). \tag{15}$$

Equations 12 and 15 are the final equations governing the currents and voltages contained within an inhomogeneous waveguide. These equations may be simplified somewhat by demanding that the loss terms be non-z dependant or even allowing them to not exist (i.e. R=G=0).

The boundary conditions to impose on Eqs. 12 and 15 are as follows:

$$\begin{array}{ll} \hat{V}(z=L,\omega) = e^{-i\beta L} & \dfrac{\partial \hat{V}(z=L,\omega)}{\partial z} = -i\beta e^{-i\beta L} \\ \hat{I}(z=L,\omega) = \dfrac{e^{-i\beta L}}{Z_{final}} & \dfrac{\partial \hat{I}(z=L,\omega)}{\partial z} = \dfrac{-i\beta e^{-i\beta L}}{Z_{final}}, \end{array} \tag{16}$$

where Z_{final} is the final characteristic impedance at the end of the line. Equation 16 implies that the voltage at port 2, or the transmitted voltage, is 1 Volt.

Additionally, the solutions to the differential equations of Eqs. 12 and 15 will not directly give the reflection and transmission coefficients. Determination of the transmission and reflection coefficients is performed by placing a homogeneous transmission line on the input and output sections of the tapered line and demanding the voltages and currents be continuous at $z = 0^-$ and $z = 0^+$, or in equation form:

$$V_0\left(e^{-i\beta z} + s_{11}e^{i\beta z}\right)\Big|_{z=0^-} = \hat{V}(z = 0^+, \omega) \quad \frac{V_0}{Z_{initial}}\left(e^{-i\beta z} - s_{11}e^{i\beta z}\right)\Big|_{z=0^-} = \hat{I}(z = 0^+, \omega),$$

where V_0, $Z_{initial}$, and s_{11} are strictly functions of frequency.

The final application of the solution to Eqs. 12 and 15 result in two equations for finding the transmission and reflection coefficients:

$$s_{11}(\omega) = \frac{\hat{V}(z = 0, \omega) - Z_{initial}\hat{I}(z = 0, \omega)}{\hat{V}(z = 0, \omega) + Z_{initial}\hat{I}(z = 0, \omega)}, \tag{17a}$$

and

$$s_{21}(\omega) = \frac{(1 + s_{11}(\omega))e^{-i\beta L}}{\hat{V}(z = 0, \omega)}\sqrt{\frac{Z_{initial}}{Z_{final}}}. \tag{17b}$$

Note that these relationships are dependant on the normalization implied in the boundary conditions of Eq. 16 imposed on the two differential equations. It is also clear from Eq. 17b that if $s_{11}(\omega)$ is known from the solution of Eq. 5, then it is not required to solve both Eq. 12 and Eq. 15 but rather either Eq. 12 or Eq. 15 to find $s_{21}(\omega)$. This has a distinct advantage if one wants to design a tapered section for a particular reflection and then later find the corresponding transmission.

c. Reflection/Transmission (FEM - finite element method)

A simple method for estimating the reflection and transmission coefficients of non-uniform transmission lines has been in practice for many years. It involves breaking up the transmission line into small segments and assuming along each particular segment exists a uniform transmission line. Figure 5 is a schematic diagram of the finite element method.

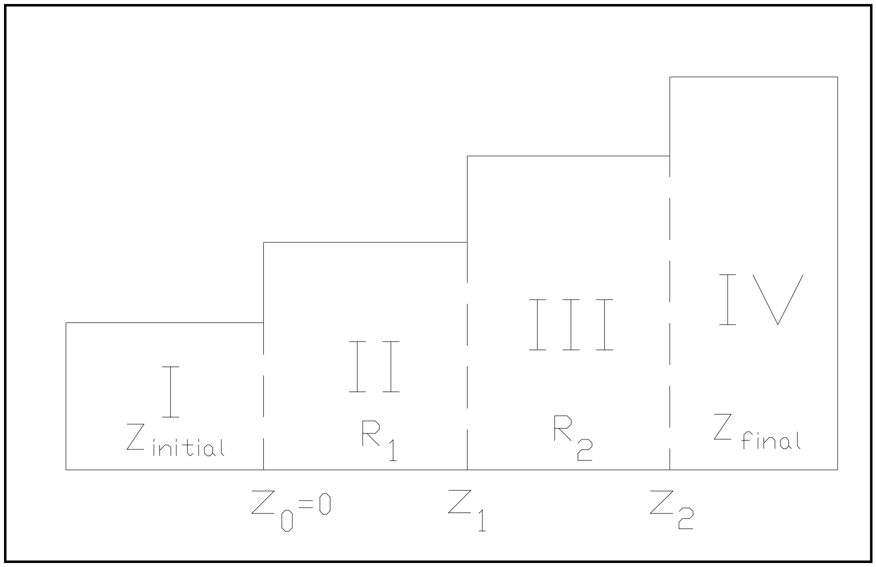

Figure 5. A schematic diagram of an inhomogeneous transmission line which is to be solved using the FEM method.

The solution to the FEM problem can be realized by using linear algebra and writing an appropriate matrix equation. The boundary condition to impose is that the reflected wave from the output element of the tapered waveguide be identically zero and the transmitted voltage wave on the input side of the tapered line have a magnitude of 1 Volt. Furthermore, to allow dispersion it is possible to make the equations general enough such that the wave number is a function of the position along the taper. For example, the first two equations are:

$$e^{-i\beta(z_0)z_0} + V_0^- e^{i\beta(z_0)z_0} = V_1^+ e^{-i\beta(\tilde{z}_0)z_0} + V_1^- e^{i\beta(\tilde{z}_0)z_0}$$
$$\frac{1}{Z_{initial}}\left(e^{-i\beta(z_0)z_0} - V_0^- e^{i\beta(z_0)z_0}\right) = \frac{1}{Z_1}\left(V_1^+ e^{-i\beta(\tilde{z}_0)z_0} - V_1^- e^{i\beta(\tilde{z}_0)z_0}\right),$$

where $\tilde{z}_m = \dfrac{z_{m+1} + z_m}{2}$. Each subsequent set of equations are as follows:

$$V_N^+ e^{-i\beta(\tilde{z}_{N-1})z_N} + V_N^- e^{i\beta(\tilde{z}_{N-1})z_N} = V_{N+1}^+ e^{-i\beta(\tilde{z}_N)z_N} + V_{N+1}^- e^{i\beta(\tilde{z}_N)z_N}$$
$$\frac{1}{Z_N}\left(V_N^+ e^{-i\beta(\tilde{z}_{N-1})z_N} - V_N^- e^{i\beta(\tilde{z}_{N-1})z_N}\right) = \frac{1}{Z_{N+1}}\left(V_{N+1}^+ e^{-i\beta(\tilde{z}_N)z_N} - V_{N+1}^- e^{i\beta(\tilde{z}_N)z_N}\right).$$

And finally, the last section's set of equations are:

$$V_M^+ e^{-i\beta(\tilde{z}_{M-1})z_M} + V_M^- e^{i\beta(\tilde{z}_{M-1})z_M} = V_{M+1}^+ e^{-i\beta(z_M)z_M}$$
$$\frac{1}{Z_M}\left(V_M^+ e^{-i\beta(\tilde{z}_{M-1})z_M} - V_M^- e^{i\beta(\tilde{z}_{M-1})z_M}\right) = \frac{1}{Z_{final}} V_{M+1}^+ e^{-i\beta(z_M)z_M}.$$

The resulting set of equations to be solved can be constructed into a large matrix equation. The form of the matrix equation is as follows:

$$\overline{\overline{A}}\vec{x} = \vec{b}. \tag{18}$$

For the case of a simple example depicted in Fig. 5, the elements of the matrix and vectors are:

$$\overline{\overline{A}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ \frac{-1}{Z_{initial}} & \frac{-1}{R_1} & \frac{1}{R_1} & 0 & 0 & 0 \\ 0 & e^{-i\beta(\tilde{z}_0)z_1} & e^{i\beta(\tilde{z}_0)z_1} & -e^{-i\beta(\tilde{z}_1)z_1} & -e^{i\beta(\tilde{z}_1)z_1} & 0 \\ 0 & \frac{e^{-i\beta(\tilde{z}_0)z_1}}{R_1} & \frac{-e^{-i\beta(\tilde{z}_0)z_1}}{R_1} & \frac{-e^{-i\beta(\tilde{z}_1)z_1}}{R_2} & \frac{e^{-i\beta(\tilde{z}_1)z_1}}{R_2} & 0 \\ 0 & 0 & 0 & e^{-i\beta(\tilde{z}_1)z_2} & e^{-i\beta(\tilde{z}_1)z_2} & e^{-i\beta(z_2)z_2} \\ 0 & 0 & 0 & \frac{e^{-i\beta(\tilde{z}_1)z_2}}{R_2} & \frac{-e^{-i\beta(\tilde{z}_1)z_2}}{R_2} & \frac{-e^{-i\beta(z_2)z_2}}{Z_{final}} \end{bmatrix},$$

and

$$\vec{x} = \begin{bmatrix} V_0^- \\ V_1^+ \\ V_1^- \\ V_2^+ \\ V_2^- \\ V_3^+ \end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} -1 \\ -\dfrac{1}{Z_{initial}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is clear from this example that $\bar{\bar{A}}$ contains information about the grid spacing and dispersion in the tapered section, $\vec{x}$ is the unknown set of variables, and $\vec{b}$ contains the boundary condition information invoked at the onset of the problem.

The reflection coefficient is simply $s_{11} = V_0^-$ and the transmission coefficient is

$$s_{21} = V_3^+ e^{-i\beta L} \sqrt{\frac{Z_{initial}}{Z_{final}}}.$$

Use of the FEM method is computationally slow and memory intensive, especially when one wants to simulate a frequency for a mode in a waveguide going through cutoff since the step size must shrink at the point of cutoff and therefore the dimension of $\bar{\bar{A}}$ increases. It is also difficult to verify the accuracy of solution without increasing the mesh and verifying that the change in transmission and reflection is below a user controllable limit or threshold. Using the FEM method is, however, both easy to set up and implement for a myriad of transmission line configurations.

III. **Numerical Solution/Examples**

Two examples will be presented in this section, namely a linearly tapered coaxial TEM line and a linearly tapered TE/TM waveguide.

a. **Coaxial TEM Example**.

A linearly tapered TEM coaxial waveguide, shown in Fig. 6, was analyzed using the three different methods outlined in this paper. The particular parameters for the tapered section are as follows:

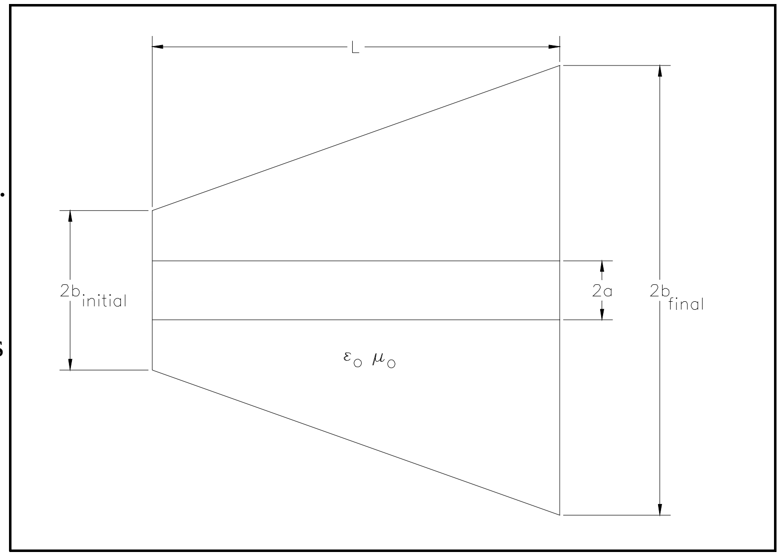

Figure 6. Schematic of a coaxial linearly-tapered transmission line. The parameters are: L=10 cm, a=1cm, $b_{initial}$=2.33 cm, and b_{final}=7 cm.

- The profile of the tapered section is: $b = \begin{cases} b_{initial} & z < 0 \\ \dfrac{b_{final} - b_{initial}}{L} z + b_{initial} & 0 \le z \le L \\ b_{final} & z > L \end{cases}$
- $b_{initial}$=2.33 cm
- b_{final}=7 cm
- L=10 cm
- a=1 cm
- The frequency range of solution is between: 50 MHz and 2 GHz with 1000 frequency points.
- Seventy equally spaced divisions were chosen for the FEM method.

The results of this simulation for reflection and transmission are shown in Figs. 7 and 8 respectively. A very close correspondence was observed between each method. The parameters for the differential equation methods are as follows:

$$Z_{line}(z) = \frac{1}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}} \ln\left(\frac{b(z)}{a}\right),$$

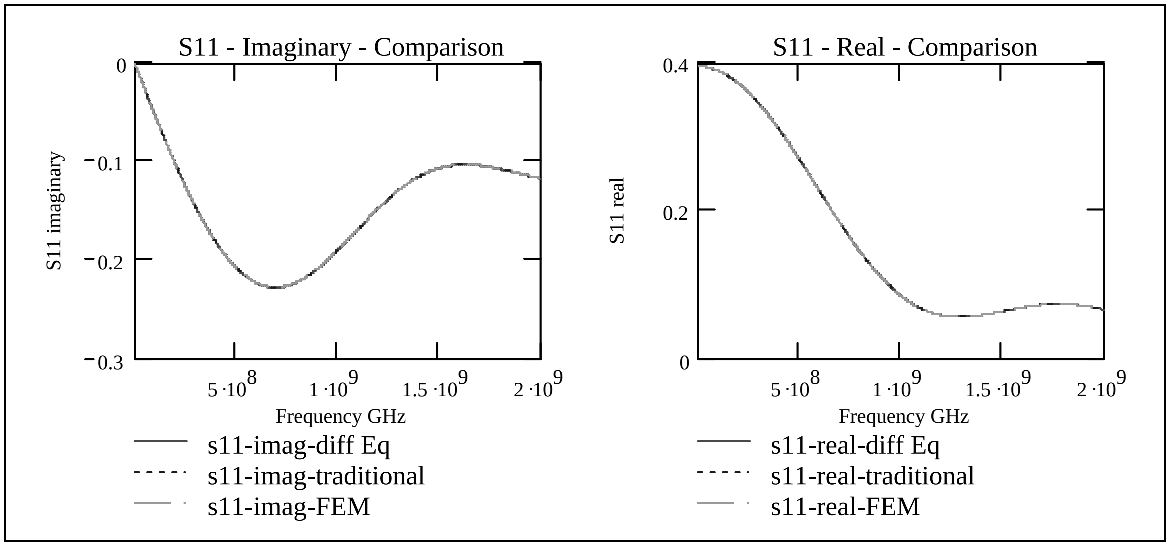

Figure 7. The calculated reflection from the TEM linear tapered coaxial transmission line depicted in Fig. 6.

$$L(z) = \frac{\mu_o}{2\pi} \ln\left(\frac{b(z)}{a}\right),$$

$$C(z) = \frac{2\pi\varepsilon_o}{\ln\left(\frac{b(z)}{a}\right)},$$

and

$$R(z) = G(z) = \frac{dR(z)}{dz} = \frac{dC(z)}{dz} = 0.$$

The solution times for the each of the differential equation methods was 1.5 minutes using the numerical differential equation solver in Mathematica version 3.0 by Wolfram Research. The solution time for the FEM method was 1.25 hours. The number of divisions was arrived at after several iterations by direct comparison of the change in iterative FEM results and with a comparison to the differential equations results.

b. TE/TM Example.

As stated in II.b, it is required to use a different differential circuit for TE and TM modes to

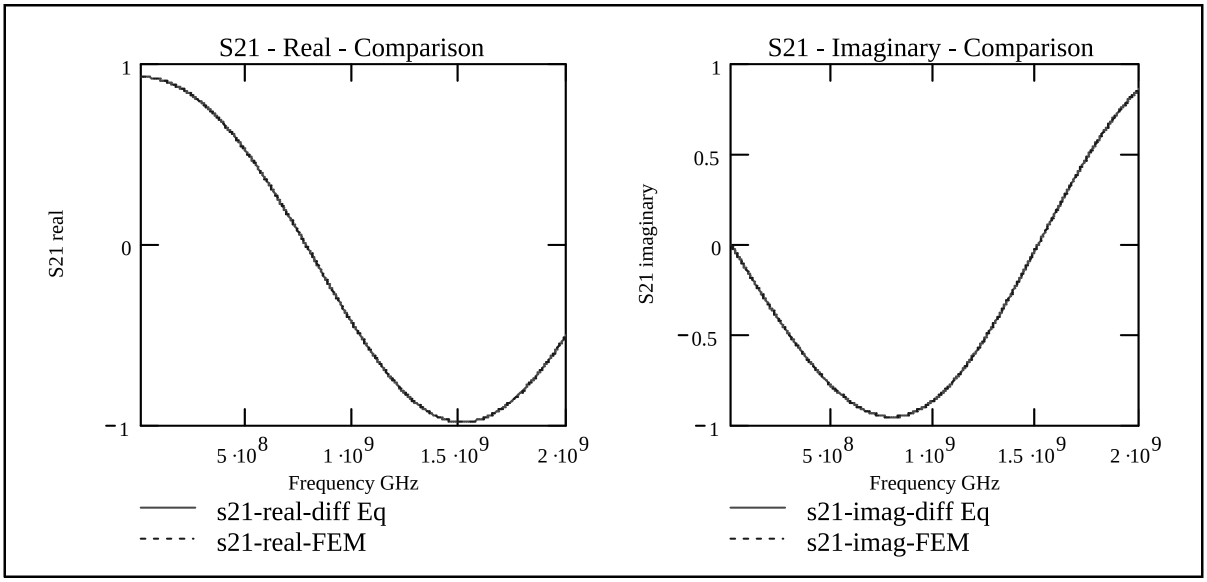

Figure 8. The calculated transmission coefficient for the TEM linear tapered coaxial transmission line depicted in Fig. 6.

accommodate the extra energy storage. The differential circuits which govern the TE and TM modes are repeated in Figs. 9 and 10 respectively [4].

For a tapered waveguide, the cutoff terms ω_c and k_c are functions of z. The method for

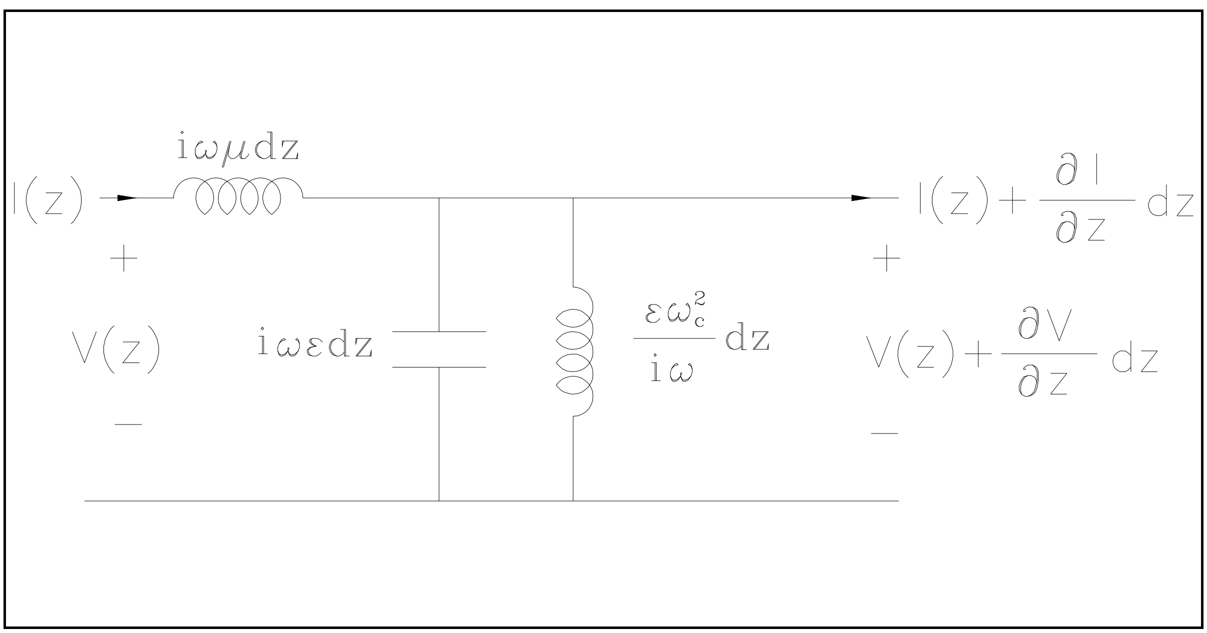

Figure 9. The schematic of a differentially small section of TE transmission line.

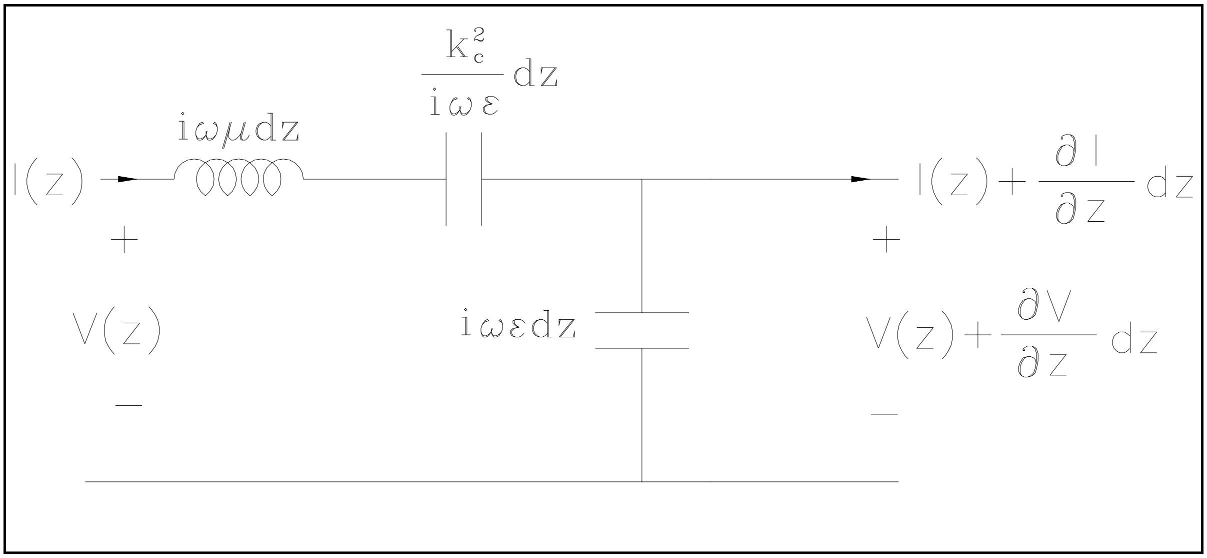

Figure 10. The schematic of a differentially small section of TM transmission line.

finding the differential equations which govern the transmission and reflection coefficients are found in the same manner as was outlined for the TEM transmission line in Sec. II.b. The resulting differential equations which govern the fields for a gradual tapered TE and TM waveguide transmission lines are shown in Eqs. 18 and 19 respectively.

$$\begin{gathered} -\frac{\partial^2 \hat{V}}{\partial z^2} = \left(\omega^2 \mu\varepsilon - \varepsilon\mu\omega_c^2\right)\hat{V} \\ -\frac{\partial^2 \hat{I}}{\partial z^2} + \frac{2\omega_c}{-\omega^2 + \omega_c^2}\frac{d\omega_c}{dz}\frac{\partial \hat{I}}{\partial z} = \left(\omega^2 \mu\varepsilon - \varepsilon\mu\omega_c^2\right)\hat{I} \end{gathered} \tag{18}$$

$$\begin{gathered} -\frac{\partial^2 \hat{I}}{\partial z^2} = \left(\omega^2 \mu\varepsilon - \varepsilon\mu\omega_c^2\right)\hat{I} \\ -\frac{\partial^2 \hat{V}}{\partial z^2} + \frac{2\omega_c}{-\omega^2 + \omega_c^2}\frac{d\omega_c}{dz}\frac{\partial \hat{V}}{\partial z} = \left(\omega^2 \mu\varepsilon - \varepsilon\mu\omega_c^2\right)\hat{V} \end{gathered} \tag{19}$$

The same boundary conditions imposed by Eq. 16 and the identical normalization required in Eqs. 17a and 17b are to be applied in the same way on the waveguide circuits of Eqs. 18 and 19..

The linearly tapered waveguide circuit of Fig. 1 was simulated using the three different methods presented in this paper. The parameters for the taper of Fig 1 are as follows:

- $a_{initial}$=27 cm
- a_{final}=20 cm
- b=10 cm
- L=3
- 600 evenly spaced frequency points between 600 MHz and 1 GHz for the $TE_{1,0}$ mode, and 600 evenly spaced frequency points between 1.6 GHz and 2 GHz for the $TM_{1,1}$ mode.
- The FEM results were produced by the HFSS program by Ansoft, Inc.
- The $TE_{1,0}$ and $TM_{1,1}$ modes were simulated.

The FEM results took 1.5 days to solve and used both magnetic and electric symmetries, which reduced the size of the matrix equation to 25% of a problem which does not take advantage of symmetry. It also only solved for 300 frequency points. The tolerances on the input and output s-parameters was set to 0.001.

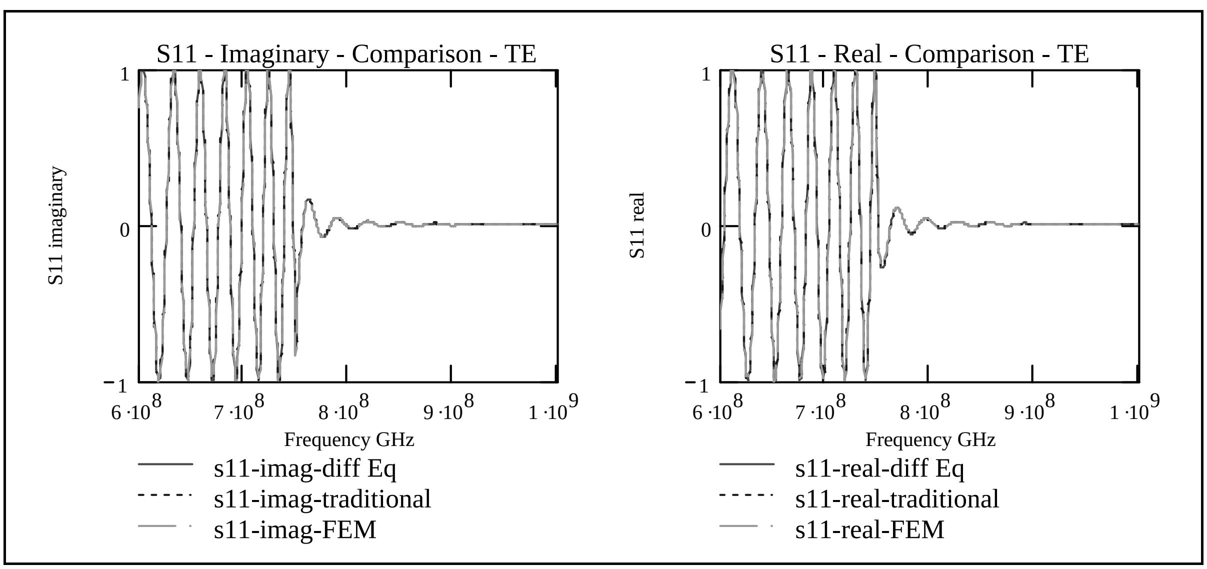

Figure 11. The $TE_{1,0}$ simulation results for the structure shown in Fig. 1.

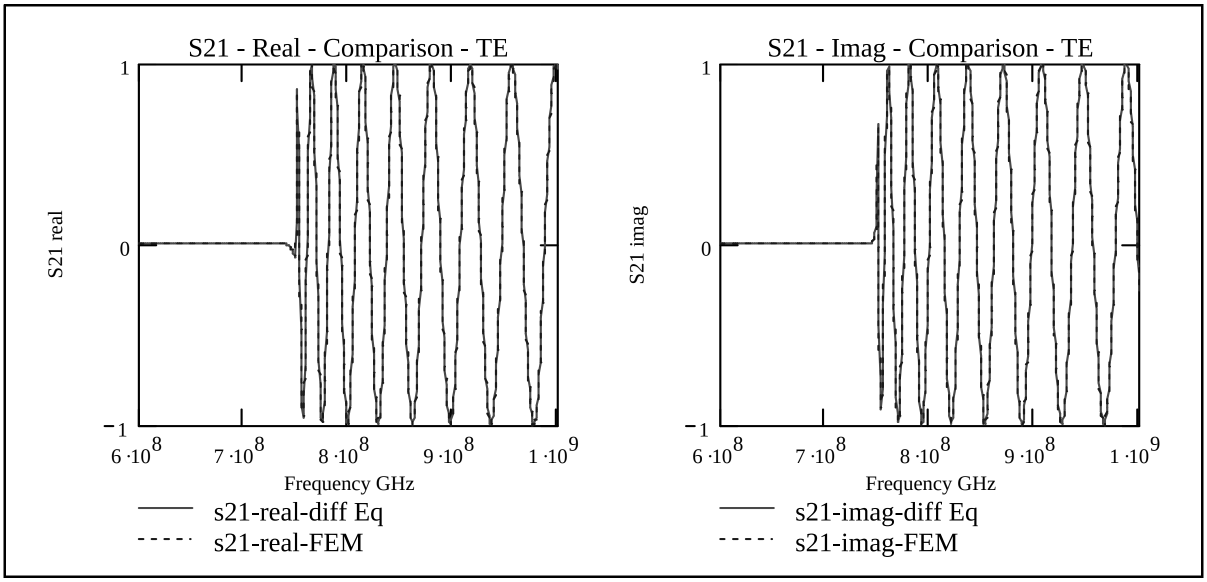

Figure 12. The $TE_{1,0}$ simulation results for the structure shown in Fig. 1.

The differential equation methods, however, took only 2.5 minutes to reach a solution, and were solved using the numerical differential equation solver from Mathematica v3.0 by Wolfram Research. The calculated transmission and reflection coefficients for the $TE_{1,0}$ mode are shown in

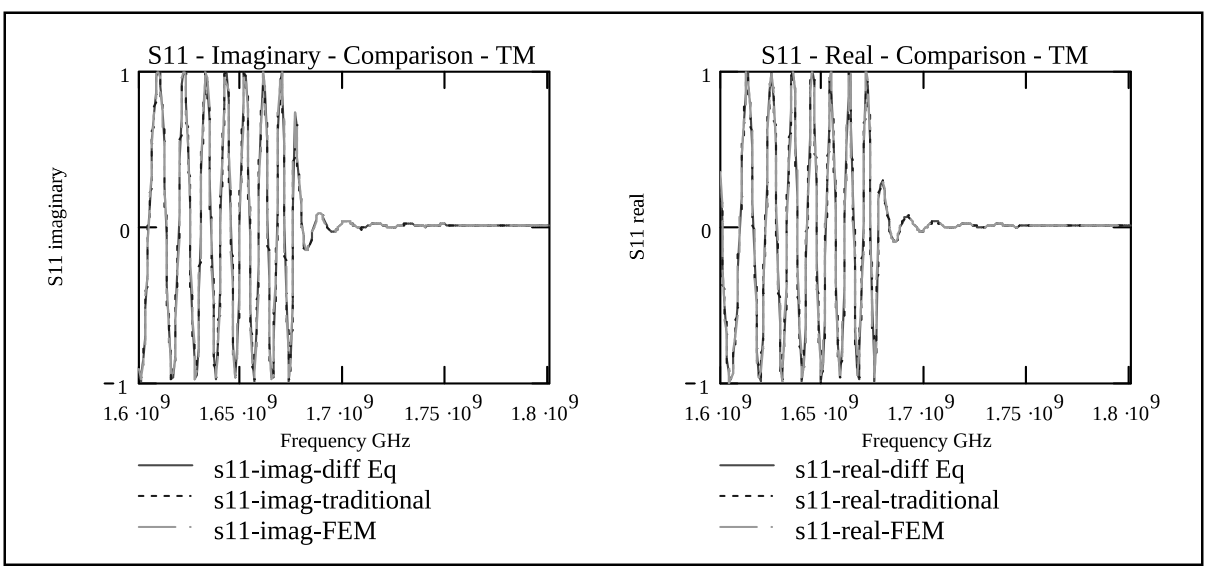

Figure 13. The $TM_{1,1}$ simulation results for the structure shown in Fig. 1.

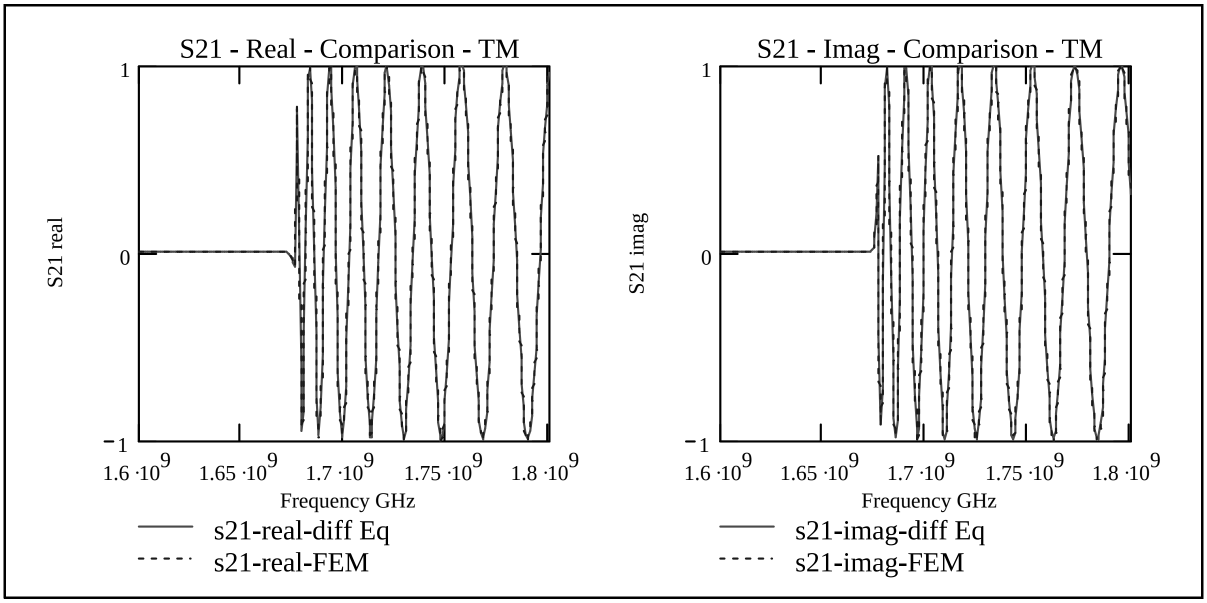

Figure 14. The $TM_{1,1}$ simulation results for the structure in Fig. 1.

Figs. 11 and 12 and the calculated reflection and transmission coefficients for the $TM_{1,1}$ mode are shown in Figs. 13 and 14. Clearly, for waveguide problems simulated at frequencies where the waveguide taper goes through cutoff, it is advantageous to use non-FEM methods to solve for the reflection and transmission coefficients to save time and computer resources. The two differential equation methods produced similar reflection results. It is important to note that choice of frequency in the simulation must be chosen such that at least one port of the waveguide is above the cutoff frequency for the mode of interest. Nonsensical results will be observed for reflection and transmission simulations when the structure is simulated at a frequency below its cutoff frequency.

IV. Conclusions

The methods in this paper show that the problem of calculating the transmission and reflection coefficients can be calculated in several different ways. The FEM method is very versatile but can be extremely memory and time intensive to solve on a computer. The differential equation method is the most general and is solved quickly on a modern computer armed with a strong numerical arsenal. The differential equation of the first section is quick to solve and can be relatively easily used in designing a tapered section for a particular desired reflection coefficient. Care must be taken while simulating waveguides because at least one port on the waveguide for a given mode must be operating above its cutoff frequency. The different methods show excellent agreement and each method has particular strengths for individual situations.

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