PBAR Note 574
Simulations of Transverse Stochastic Cooling using the Fokker-Planck Equation
** DRAFT **

John Marriner

March 16, 1998

This memo is written as a note for some one who already understands stochastic cooling and wants to understand the mathematical basis for the computer code used in the simulation.

SYSTEM GAIN FUNCTION

The system gain G is defined as

$$\Delta x = 2Gx \qquad [1]$$

where Dx = the *average* change in "amplitude squared" per turn per Schottky band. The total change is given by a summation over Schottky bands. The "amplitude squared" is measured in units of emittance, namely, m-rad. Equation [1] includes a factor of 1/2 from averaging over $\cos^2 f$, where f is the phase of the betatron motion.

The system gain is the product of the pickup response (p), the electronic gain (g), and the kicker response (k).

$$G = p \bullet g \bullet k \qquad [2]$$

The pickup response is defined so that the voltage into a 1Ω system produced by a particle with amplitude squared x per Schottky band is

$$V_p = \sqrt{2}\, p \cdot \sqrt{x} \qquad [3]$$

The rms voltage produced is

$$V_p^{rms} = p \cdot \sqrt{x} \qquad [4]$$

Define the kicker voltage (also in a1Ω system) as

$$\begin{aligned} V_k &= gV_p \\ &= \sqrt{2}\, pg \cdot \sqrt{x} \end{aligned} \qquad [5]$$

It follows from [1] through [5] that the kicker response *k* is given by

$$\Delta x = \sqrt{2} V_k k \sqrt{x} \qquad [6]$$

The kick *k* is equivalently defined as :

$$\Delta q = \frac{1}{2\sqrt{2}} \frac{kV_k}{b^2 \; E/e} \qquad [7]$$

because
****

$$x = b_y q^2 \qquad []$$

and

$$\Delta x = 2bq\Delta q \qquad []$$

In addition G includes the bad mixing and phase advance factors which are
****

$$phase = e^{\pm ia_2 Q} xb$$

SIGNAL SUPPRESSION FACTOR

The beam response if given by

$$\mathrm{F} = \frac{1}{2} \int e^{2pia_1 f(1/f_0 - T)} \left\{ e^{ia_2 Q} \left[1 + \frac{i}{\tan(pf/f_0 + Q)} \right] + e^{ia_2 Q} \left[1 + \frac{i}{\tan(pf/f_0 + Q)} \right] \right\} \Psi(f_0) df_0$$

and the gain function is reduced by the factor *d*, where

$$d = 1 + \mathrm{F} G .$$

FOKKER-PLANCK EQUATION

The Fokker-Planck equation is

$$\begin{aligned} \frac{\P\Psi}{\P t} &= -\frac{\P\Phi}{\P x} \\ &= -\frac{\P}{\P x}\left[F\Psi - \left(D_2 + D_1 \langle e \rangle\right) \frac{\P\Psi}{\P e} \right] \end{aligned} \qquad [8]$$

where Y is the particle density per unit *x* and Φ is the flux of particles (number per unit time) crossing a given value of *x*.

The boundary conditions are

$$\Phi(0,t) = 0 \qquad [9]$$

and

$$\Psi(x_{\max}, t) = 0 \qquad [10]$$

The first condition is that the flux of particles into $x<0$ is zero. The second is that there is a hard aperture at $x = x_{\max}$ and that the particle density is zero for $x \geq x_{\max}$.

The first moment of the Fokker-Planck equation results in the standard cooling formula. Specifically,

$$\begin{aligned} \int x \frac{\partial \Psi}{\partial t} dx &= -\int x \frac{\partial \Phi}{\partial x} dx \\ &= -x_{\max} \Phi(x_{\max}) + \int \Phi dx \end{aligned} \qquad [11]$$

We define the rate of the decrease in emittance as

$$\frac{1}{t_x} = \frac{1}{\int x \Psi dx} \left[x_{\max} \Phi(x_{\max}) - \int \Phi dx \right] \qquad [12]$$

where the first term in the brackets on the right hand side is the emittance decrease due to particle loss and the second term is the cooling effect.

Substituting Equation [9] into Equation [12], assuming that the flux vanishes at $x_{\max}$,

$$\int x \frac{\partial \Psi}{\partial t} dx = \int \left[F\Psi - \left(D_2 + D_1 \langle x \rangle\right) \frac{\partial \Psi}{\partial x} \right] dx \qquad [13]$$

Assuming that F, D_1, and D_2 are linear functions of x, we obtain

$$\int x \frac{\partial \Psi}{\partial t} dx = \int \left[F_0 x \Psi - \left(D_{20} x + D_{10} x \langle x \rangle\right) \frac{\partial \Psi}{\partial x} \right] dx \qquad [14]$$

which can be rewritten as

$$N \frac{d\langle x \rangle}{dt} = F_0 \langle x \rangle N + D_{20} N + D_{10} \langle x \rangle N \qquad [15]$$

or equivalently

$$\frac{d\langle x\rangle}{dt} = F_0\langle x\rangle + D_{20} + D_{10}\langle x\rangle \qquad [16]$$

where

$$N = \int_0^{x_{\max}} \Psi(x)dx \qquad [17]$$

and

$$\langle x\rangle = \frac{1}{N}\int_0^{x_{\max}} x\Psi(x)dx \qquad [18]$$

The coefficients are given by

$$F_0 = 2\,\mathrm{Re}\left[-\frac{iG}{d}e^{\pm i a_2 Q}e^{-if_k}\right]f_0 \qquad [19]$$

where the ± is for the n±Q sidebands, the pickup to kicker phase advance is a_2 times the tune, and f_k is the "bad" mixing between pickup and kicker.

$$f_k = \frac{2p\,a_1 d\Delta E(n\pm Q)h}{b^2 E}$$

$$D_{10} = \left|\frac{G}{d}\right|^2 f_0\,(2M) \qquad [20]$$

$$D_{20} = k_B T_N |gk|^2 f_0^2 \qquad [18]$$

$$2M = \frac{1}{N}\frac{d\Psi}{df}\frac{f_0}{n\pm Q} \qquad [19]$$

Summing [14] over $2W/f_0$ Schottky bands - assuming that G/d, M, *etc.* are constant

$$\frac{d\langle x\rangle}{dt} = W\left[-4G + 4G^2 N(M+U)\right] \qquad [20]$$

where

$$2U = \frac{k_B T_N}{p^2} \qquad [21]$$

Substitution of $G' = 2GN$ results in the standard form.

OPTIMUM GAIN CALCULATION

The optimum gain is determined by the condition that the LHS of Equation [14] is a minimum. We define a new gain $g' = gs$ and desire to find the value of s such that g' is the optimum gain. We require F and D and their derivatives as functions of s

$$F = \frac{Gs}{1+\mathrm{F}Gs} + c.c. \qquad [22]$$

$$\frac{\P F}{\P s} = \frac{G(1+\mathrm{F}Gs) - Gs(\mathrm{F}G)}{(1+\mathrm{F}Gs)^2} + c.c.$$

$$= \left(\frac{G}{(1+\mathrm{F}Gs)^2}\right) + c.c. \qquad [23]$$

$$\frac{\P^2 F}{\P s^2} = \frac{-2\mathrm{F}G^2}{(1+\mathrm{F}Gs)^3} + c.c. \qquad [24]$$

$$D = \frac{G^* G s^2}{(1+\mathrm{F}Gs)(1+\mathrm{F}^* G^* s)}$$

$$= \frac{|G|^2 s^2}{1+|\mathrm{F}|^2 |G|^2 s^2 + (\mathrm{F}G + \mathrm{F}^* G^*)s} \qquad [25]$$

$$\frac{\P D}{\P s} = \frac{|G|^2 (2s + \mathrm{F}G + \mathrm{F}^* G^*) + 2s|G|^2}{|1+\mathrm{F}Gs|^4} \qquad [26]$$

$$\frac{\P^2 D}{\P s^2} = \frac{2|G|^2 \left\{ |1+\mathrm{F}Gs|^2 + (2s + \mathrm{F}G + \mathrm{F}^* G^*)\left[2|\mathrm{F}|^2 |G|^2 s + (\mathrm{F}G + \mathrm{F}^* G^*)\right]\right\}}{|1+\mathrm{F}Gs|^6} \qquad [27]$$

The equation

$$\left\langle \frac{d\Delta A^2}{dt} \right\rangle = \frac{f_o A^2}{2N} \sum_{\pm n \pm Q > 0} \sum \left\{ -2\mathrm{Re}\left[i\tilde{G}\left((n \pm Q)\omega_i\right) e^{i\mu(s)} e^{-i(n \pm Q)\omega_i t_{ik}} \right] \right.$$

$$+ \left[M\left((n \pm Q)\omega_i \right) + U\left((n \pm Q)\omega_i \right) \right] \left| \tilde{G}\left((n \pm Q)\omega_i \right) \right|^2 \Bigg\} \qquad [8.28]$$

$$F(\omega) = \frac{\sqrt{\beta(s)\beta(0)}}{4} \int_0 \Psi(\omega_o)\, e^{-i\omega s/v} \left\{ e^{-i\mu(s)} \left[\frac{1}{\tan \pi \left(\frac{\omega}{\omega_o} + Q \right)} + i \right] \right.$$

$$\left. - e^{+i\mu(s)} \left[\frac{1}{\tan \pi \left(\frac{\omega}{\omega_o} - Q \right)} + i \right] \right\} d\omega_o \,. \qquad [9.14]$$