IONIZATION COOLING IN THE CASE OF STRONG LONGITUDINAL-TRANSVERSE COUPLING

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FNAL

Emittance Exchange Workshop,
LBNL, October 2001

Talk Outline

1. Introduction
2. Twiss parameters in the case of strongly coupled motion
3. Cooling decrements
4. Transfer matrix parameterization
5. Motion stability and ionization cooling
6. Numerical example
7. Conclusions

http://cosmo.fnal.gov/organizationalchart/lebedev/NeutrinoFactory/NuFact.html

1. Introduction

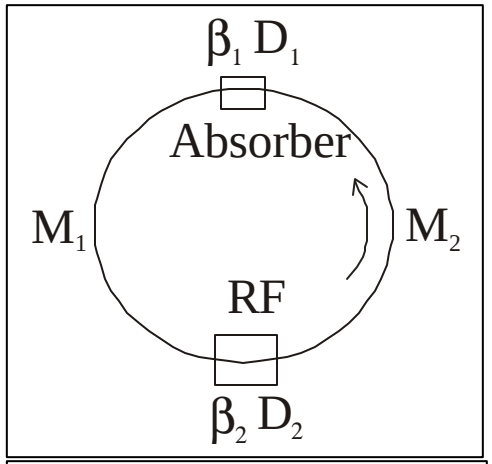
- Emittance exchange
  - Non-zero dispersion
- High voltage and non-zero dispersion
  - Strong coupling
- Linear analytical theory can be extremely useful supplement to numerical simulations
  - Motion symplecticity
  - Motion symmetry
  - Cooling length is much longer than one period/revolution
    - Perturbation theory is used to find damping
- Both linear channel and ring are described by the same theory

2. Twiss Parameters in the Case of Strongly Coupled Motion

$$\mathbf{M}_{total} = \mathbf{M}_{RF}\mathbf{M}_1\mathbf{M}_{wedge}\mathbf{M}_2$$

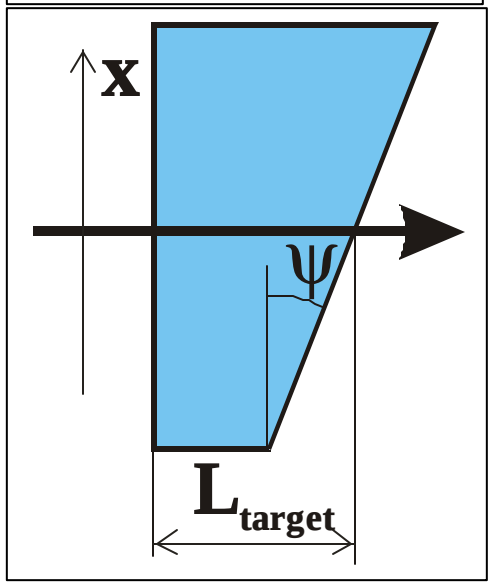
Matrices are written relative to the reference particle having no betatron and synchrotron motion

$$\mathbf{x} \equiv \begin{bmatrix} x \\ q_x \\ s \\ \Delta p/p \end{bmatrix}$$



<u>Cooling and wedge transfer matrices</u>

$$\mathbf{M}_{RF} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k_s & 1 \end{bmatrix}, \qquad \mathbf{M}_{wedge} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -d_p & 0 & 0 & 1 \end{bmatrix}$$



Here: $d_0 = \left.\dfrac{\Delta p}{p}\right|_{x=0}$ - momentum loss for particle coming through the absorber center

$k_s = \dfrac{2pV_{RF}\cos f}{pb_v l_{RF}}$ - longitudinal focusing strength

$d_p = d_0 \dfrac{\tan y}{L_{target}}$ - wedge coupling

Symplecticity of the Eigen-vectors[1]

For a single-particle phase-space trajectory along the orbit is

$$\mathbf{x}(s) = \mathrm{Re}\left(\sqrt{e_1}\,\mathbf{v}_1(s)e^{-i(y_1+m_1(s))} + \sqrt{e_2}\,\mathbf{v}_2(s)e^{-i(y_2+m_2(s))}\right)$$

- $\mathbf{v}_1(s) = \begin{bmatrix} \sqrt{b_{1x}(s)} \\ -\dfrac{i(1-u(s))+a_{1x}(s)}{\sqrt{b_{1x}(s)}} \\ \sqrt{b_{1y}(s)}e^{in_1(s)} \\ -\dfrac{iu(s)+a_{1y}(s)}{\sqrt{b_{1y}(s)}}e^{in_1(s)} \end{bmatrix}$, $\mathbf{v}_2(s) = \begin{bmatrix} \sqrt{b_{2x}(s)}e^{in_2(s)} \\ -\dfrac{iu(s)+a_{2x}(s)}{\sqrt{b_{2x}(s)}}e^{in_2(s)} \\ \sqrt{b_{2y}(s)} \\ -\dfrac{i(1-u(s))+a_{2y}(s)}{\sqrt{b_{2y}(s)}} \end{bmatrix}$

  are the eigen-vectors at coordinate s
  - y_1 and y_2 are the initial phases of betatron motion;
  - and $\sqrt{e_1} \equiv I_1$ and $\sqrt{e_2} \equiv I_2$ are the actions

- Only 8 of 11 parameters are independent
  - Usually $u(s)$, $n_1(s)$ and $n_2(s)$ can be computed through b's and a's

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[1] **BETATRON MOTION WITH COUPLING OF HORIZONTAL AND VERTICAL DEGREES OF FREEDOM**, V. Lebedev, S. Bogacz, **JLab-TN-00-022 ,** <http://tnweb.jlab.org/tn/2000/>

- In the mirror symmetry case, $a_{1x} = a_{1y} = a_{2x} = a_{2y} = 0$, the symplecticity conditions leave four independent parameters (b_x, b_y, u and k) for eigen vector parameterization

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{b_x} \\ -i\dfrac{1-u}{\sqrt{b_x}} \\ \pm i\sqrt{b_y} \\ \pm\dfrac{u}{\sqrt{b_y}} \end{bmatrix}, \quad \mathbf{v}_2(s) = \begin{bmatrix} \pm ik\sqrt{b_x} \\ \pm k^{-1}\dfrac{u}{\sqrt{b_x}} \\ k^{-1}\sqrt{b_y} \\ -ik\dfrac{1-u}{\sqrt{b_y}} \end{bmatrix},$$

or

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{b_x} \\ -i\dfrac{1-u}{\sqrt{b_x}} \\ \pm\sqrt{b_y} \\ \mp i\dfrac{u}{\sqrt{b_y}} \end{bmatrix}, \quad \mathbf{v}_2(s) = \begin{bmatrix} \mp k\sqrt{b_x} \\ \pm ik^{-1}\dfrac{u}{\sqrt{b_x}} \\ \sqrt{b_{y2}} \\ -i\dfrac{u}{\sqrt{b_{y2}}} \end{bmatrix}, \text{ where } \sqrt{b_{y2}} = k^{-1}\frac{1-u}{u}\sqrt{b_y}$$

- The choice of one of these two representations is determined by transfer matrix

Coming from the imaginary to real representation of turn-by-turn positions

$$\boxed{\mathbf{x}_n = \mathbf{V}\mathbf{a}_n}$$

where

$$\mathbf{V} = \left[\mathbf{v}_1', -\mathbf{v}_1'', \mathbf{v}_2', -\mathbf{v}_2''\right] = \begin{bmatrix} \sqrt{b_x} & 0 & 0 & \mp k\sqrt{b_x} \\ 0 & \dfrac{1-u}{\sqrt{b_x}} & \pm k^{-1}\dfrac{u}{\sqrt{b_x}} & 0 \\ 0 & \mp\sqrt{b_y} & k^{-1}\sqrt{b_y} & 0 \\ \pm\dfrac{u}{\sqrt{b_y}} & 0 & 0 & k\dfrac{1-u}{\sqrt{b_y}} \end{bmatrix}$$

- is the symplectic eigen-vector matrix

$$\mathbf{a}_n = \begin{bmatrix} \sqrt{e_1}\cos(y_1 + nm_1) \\ \sqrt{e_1}\sin(y_1 + nm_1) \\ \sqrt{e_2}\cos(y_2 + nm_2) \\ \sqrt{e_2}\sin(y_2 + nm_2) \end{bmatrix}$$

- is the amplitude vector

3. Cooling Decrements

Cooling due to passing through absorber

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

or for amplitudes

$$\mathbf{V}\mathbf{a}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{V}\mathbf{a} \Rightarrow \quad \mathbf{a}' = \mathbf{V}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{V}\mathbf{a}$$

Using following expressions:

$$\mathbf{V}^{-1} = -\mathbf{U}\mathbf{V}^{T}\mathbf{U} = \begin{bmatrix} \frac{1-u}{\sqrt{b_x}} & 0 & 0 & \pm\sqrt{b_y} \\ 0 & \sqrt{b_x} & \mp\frac{u}{\sqrt{b_y}} & 0 \\ 0 & \pm k\sqrt{b_x} & k\frac{1-u}{\sqrt{b_y}} & 0 \\ \mp k^{-1}\frac{u}{\sqrt{b_x}} & 0 & 0 & k^{-1}\sqrt{b_y} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{V}^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-u & \pm k^{-1}u & 0 \\ 0 & \pm k^{-1}(1-u) & u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We obtain

$$\mathbf{a}' = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - d \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-u & \pm k^{-1}u & 0 \\ 0 & \pm k^{-1}(1-u) & u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \mathbf{a}$$

- Calculating values of the actions after cooling one obtains

$$e_1' \equiv a_1'^2 + a_2'^2 = e_1\left[1 - 2d_0(1-u)\sin^2 y_1\right] \pm 2\sqrt{e_1 e_2}\, d_0 u \sin y_1 \cos y_2 + O(d^2)$$

$$e_2' \equiv a_3'^2 + a_4'^2 = e_2\left[1 - 2d_0 \cos^2 y_2\right] \pm 2\sqrt{e_1 e_2}\, d_0 (1-u) \sin y_1 \cos y_2 + O(d^2)$$

- Averaging over betatron phases finally yields the damping per turn

$$\frac{de_1}{e_1} = -d_0(1-u)$$

$$\frac{de_2}{e_2} = -d_0 u$$

Cooling due to passing through wedge

$$\mathbf{a}' = \mathbf{V}^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -d_p & 0 & 0 & 1 \end{bmatrix}\mathbf{V}\mathbf{a} = \mathbf{a} - d_p\sqrt{b_x b_y}\begin{bmatrix} 1 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k^{-1} & 0 & 0 & -1 \end{bmatrix}\mathbf{a}$$

- where matrix **V** is taken at the wedge

- Performing similar calculations after averaging over betatron phase we have

$$\frac{de_1}{e_1} = -d_p\sqrt{b_x b_y} \quad , \qquad \frac{de_2}{e_2} = d_p\sqrt{b_x b_y}$$

- Combining damping affects from wedge and absorber we finally obtain

$$\frac{de_1}{e_1} = -d_0(1-u)\Big|_{cavity} - d_p\sqrt{b_x b_y}\Big|_{absorber}$$

$$\frac{de_2}{e_2} = -d_0 u\Big|_{cavity} + d_p\sqrt{b_x b_y}\Big|_{absorber}$$

4. Transfer Matrix Parameterization

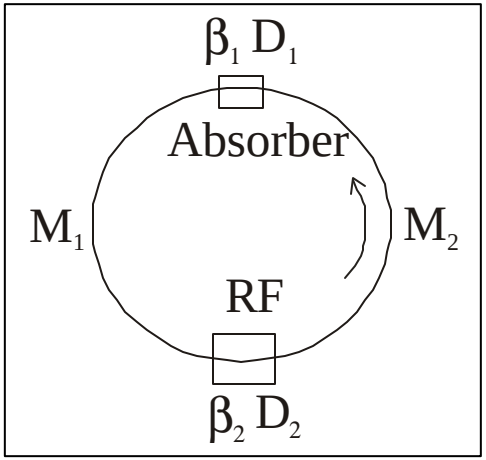
- If $\alpha_x = \alpha_y = 0$ the symplecticity of the transfer matrix yields for a half ring:

$$\mathbf{M}_1 = \begin{bmatrix} \sqrt{\frac{b_2}{b_1}}\cos\frac{m}{2} & \sqrt{b_1 b_2}\sin\frac{m}{2} & 0 & D_2 - D_1\sqrt{\frac{b_2}{b_1}}\cos\frac{m}{2} \\ -\frac{1}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & \sqrt{\frac{b_1}{b_2}}\cos\frac{m}{2} & 0 & \frac{D_1}{\sqrt{b_1 b_2}}\sin\frac{m}{2} \\ -\frac{D_2}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & D_2\sqrt{\frac{b_1}{b_2}}\cos\frac{m}{2} - D_1 & 1 & -m_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{x} \equiv \begin{bmatrix} x \\ q_x \\ s \\ \Delta p/p \end{bmatrix}$$

Expression for $\mathbf{M}_2$ can be obtain by index exchange:

$$b_1, D_1 \Leftrightarrow b_2, D_2$$

Here $m_{56} = M_{56} - \frac{C}{2g^2}$ and C is the ring circumference



- For the entire ring (from cavity to cavity) it yields:

$$\mathbf{M} = \mathbf{M}_1 \mathbf{M}_2 = \begin{bmatrix} \cos m & b_2 \sin m & 0 & D_2(1-\cos m) \\ -\frac{1}{b_2}\sin m & \cos m & 0 & \frac{D_2}{b_2}\sin m \\ -\frac{D_2}{b_2}\sin m & -D_2(1-\cos m) & 1 & -m_{T56} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$m_{T56} = 2\left(m_{56} - \left(D_2 \cos\frac{m}{2} - D_1\sqrt{\frac{b_2}{b_1}} \right)\frac{D_2}{b_2}\sin\frac{m}{2} \right)$$

Total Transfer Matrix of the ring (no damping, from cavity to cavity)

$$\mathbf{M}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{k_s}{2}&1\end{bmatrix}\begin{bmatrix}\cos m & b_2\sin m & 0 & D_2(1-\cos m)\\ -\frac{1}{b_2}\sin m & \cos m & 0 & \frac{D_2}{b_2}\sin m\\ -\frac{D_2}{b_2}\sin m & -D_2(1-\cos m) & 1 & -m_{T56}\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{k_s}{2}&1\end{bmatrix}=$$

$$\begin{bmatrix}\cos m & b_2\sin m & \frac{2D_2}{b_s}\sin^2\frac{m}{2}\tan\frac{m_s}{2} & 2D_2\sin^2\frac{m}{2}\\ -\frac{1}{b_2}\sin m & \cos m & \frac{D_2}{b_2 b_s}\sin m\tan\frac{m_s}{2} & \frac{D_2}{b_2}\sin m\\ -\frac{D_2}{b_2}\sin m & -2D_2\sin^2\frac{m}{2} & \cos m_s & -b_s\sin m_s\\ \frac{D_2}{b_2 b_s}\sin m\tan\frac{m_s}{2} & -\frac{2D_2}{b_s}\sin^2\frac{m}{2}\tan\frac{m_s}{2} & \frac{\sin m_s}{b_s} & \cos m_s\end{bmatrix},\quad \cos m_s=1-\frac{m_{T56}k_s}{2},\quad b_s=\frac{m_{T56}}{\sin m_s}$$

The motion is uncoupled at cavity position if dispersion in the cavity, D_2, is equal to zero.

Total Transfer Matrix of the ring (no damping, from absorber to absorber)

$$\mathbf{M} = \mathbf{M}_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k_s & 1 \end{bmatrix} \mathbf{M}_1 = \begin{bmatrix} \cos m & b_1 \sin m & 0 & D_1(1-\cos m) \\ -\frac{1}{b_1}\sin m & \cos m & 0 & \frac{D_1}{b_1}\sin m \\ -\frac{D_1}{b_1}\sin m & -D_1(1-\cos m) & 1 & -m_{A56} \\ 0 & 0 & 0 & 1 \end{bmatrix} +$$

$$k_s \begin{bmatrix} D_{eff}\frac{D_2}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & (D_1 - D_{eff})\sqrt{\frac{b_1}{b_2}}D_2\cos\frac{m}{2} - D_1^2 & -D_{eff} & m_{56}D_{eff} \\ -\frac{D_2^2}{b_1 b_2}\sin^2\frac{m}{2} & D_{eff}\frac{D_2}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & \frac{D_2}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & \frac{D_2 m_{56}}{\sqrt{b_1 b_2}}\sin\frac{m}{2} \\ \frac{D_2 m_{56}}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & -m_{56}D_{eff} & -m_{56} & m_{56}^2 \\ -\frac{D_2}{\sqrt{b_1 b_2}}\sin\frac{m}{2} & D_{eff} & 1 & -m_{56} \end{bmatrix}$$

$$m_{A56} = 2\left(m_{56} - \left(D_1\cos\frac{m}{2} - D_2\sqrt{\frac{b_1}{b_2}} \right)\frac{D_1}{b_1}\sin\frac{m}{2} \right), \quad D_{eff} = D_2\sqrt{\frac{b_1}{b_2}}\cos\frac{m}{2} - D_1$$

5. Motion Stability and Ionization Cooling

Equations for eigen-vectors at RF cavity location

$$(\mathbf{M} - l_1\mathbf{I})\mathbf{v}_1 = 0 \qquad (1)$$

$$(\mathbf{M} - l_2\mathbf{I})\mathbf{v}_2 = 0 \qquad (2)$$

Knowing the form of the eigen-vectors allows one to find them easier.

- Using real parts of top and bottom equations of Eq. (1) we have

$$\begin{cases} (M_{11} - l')\sqrt{b_x} + M_{14}\dfrac{u}{\sqrt{b_y}} = 0 \\ M_{41}\sqrt{b_x} + (M_{44} - l')\dfrac{u}{\sqrt{b_y}} = 0 \end{cases} \qquad \text{where } l = l' + i\,l''$$

- The solution yields eigen-frequencies:

$$\boxed{\cos m_{1,2} \equiv l'_{1,2} = \frac{\cos m + \cos m_s}{2} \pm \sqrt{\left(\frac{\cos m - \cos m_s}{2}\right)^2 - \frac{4D_2^{\,2}}{b_2 b_s}\sin^3\frac{m}{2}\cos\frac{m}{2}\tan\frac{m_s}{2}}}$$

- Condition of the motion stability, $\left|l'_{1,2}\right| \le 1$, is determined by the betatron and synchrotron tunes and by coupling parameter - $D_2^{\,2}/b_2 b_s$

- Using real parts of top equations of both Eqs. (1) and (2) we have

$$\begin{cases} (M_{11} - l_1')\sqrt{b_x} + M_{14}\dfrac{u}{\sqrt{b_y}} = 0 \\ (M_{11} - l_2')\sqrt{b_x} - M_{14}\dfrac{u}{\sqrt{b_y}} = 0 \end{cases}$$

- Subtracting one equation from another we obtain

$$\sqrt{b_x b_y} = \frac{M_{14}}{l_1' - l_2'} = D_2\sqrt{\frac{b_1}{b_2}}\cos\frac{m}{2} - D_1 \qquad (3)$$

- Similar for the bottom equations

$$\begin{cases} M_{41}\sqrt{b_x} + (M_{44} - l_1')\dfrac{u}{\sqrt{b_y}} = 0 \\ M_{41}\sqrt{b_x} - (M_{44} - l_2')\dfrac{1-u}{\sqrt{b_y}} = 0 \end{cases}$$

- That finally yields the coupling strength

$$\boxed{u = M_{41}\frac{\sqrt{b_x b_y}}{l_1' - M_{41}} = \frac{4D_2^{\,2}}{b_2 b_s (l_1' - l_2')(\cos m_s - l_1')}\sin^3\frac{m}{2}\cos\frac{m}{2}\tan\frac{m_s}{2}}$$

Equations for eigen-vectors at absorber location

- To find damping decrements due to wedge we need to compute $\sqrt{b_x b_y}$ at absorber location.
- Using Eq.(3) we obtain

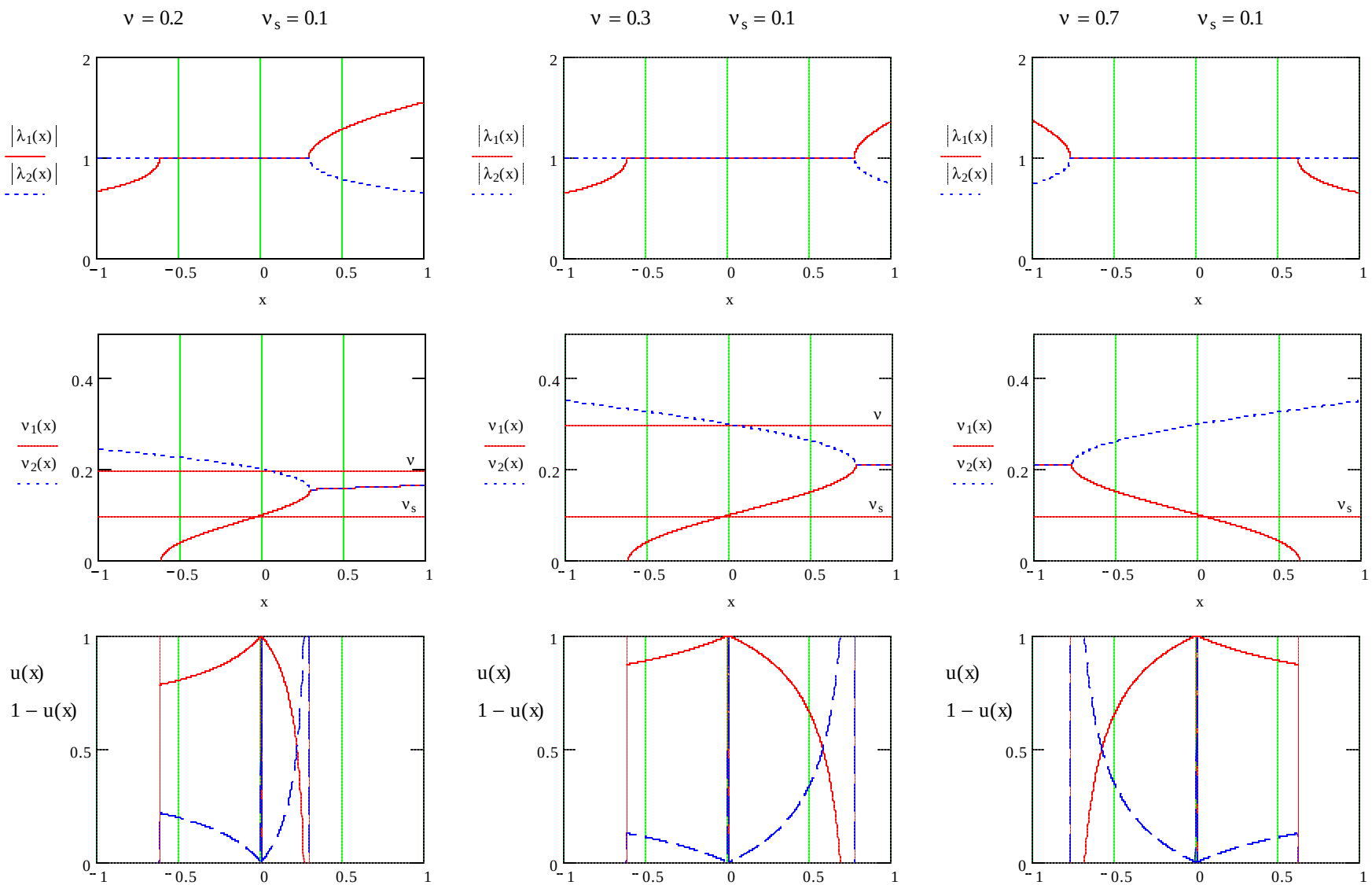
$$\sqrt{b_x b_y}\Big|_{absorber} = \frac{M_{14}}{l_1' - l_2'} = \left(2D_1 \sin^2\frac{m}{2} + k_s m_{56}\left(D_2\sqrt{\frac{b_1}{b_2}}\cos\frac{m}{2} - D_1\right)\right)\frac{1}{l_1' - l_2'}$$

- If dispersion at the cavity is equal to zero ($D_2 = 0$) then

$$\sqrt{b_x b_y}\Big|_{absorber} = -D_1$$

6. Numerical example

Energy	1 GeV
Circumference	25 m
Absorber length	40 cm
RF voltage	20 MeV
Energy loss per turn	11.2 MeV
Accelerating phase	34 deg
RF wave length	150 cm
Ring M_{56}	23 m
Synchrotron frequency	0.207
Longitudinal beta-function	24 m
Coupling parameter , $D_2^{\,2} / b_2 b_s$	0.4



Module of the eigen vector, increment per turn and coupling weight *u* as function of coupling parameter - $x = D_2^{\,2} / b_2 b_s$

7. Conclusions

- Dispersion in the cavity is equal to zero
  - There is no transverse-longitudinal coupling
    - Horizontal and longitudinal tunes are decoupled
    - Eigen vectors at cavity (not everywhere) are decaupled
  - Not wedged absorber
    - Only transverse damping is present
  - Wedged absorber
    - allows one to transfer part of transverse damping to longitudinal motion
      - Too large wedge or its wrong sign make the motion unstable
    - Normally damping is sufficiently small $n'' \equiv \frac{1}{2p}\frac{de}{e} \approx 10^{-3}$
      - Absorber does not produce as strong coupling as cavity $n' \approx 0.1$

- Dispersion in the cavity is not zero
  - Coupling between horizontal and longitudinal degrees of freedom can make motion unstable
  - Coupling value is determined by three parameters
    - Horizontal tune
    - Longitudinal tune
    - Coupling parameter - $D_2^{\,2} / b_2 b_s$
  - Strong dependence of both tunes on the dispersion
  - Adjusting coupling one can redistribute decrements in desired proportion
  - Using wedged absorber can be useful but is not required