IONIZATION COOLING IN AXIAL SYMMETRIC CHANNEL

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Talk Outline

1. Twiss Parameters and Beam Emittances
2. Cooling Description
3. Canonical Angular Momentum and Second Order Moments
4. Beta-functions for Particle Motion with Axial-symmetric Solenoidal Focusing

1. Twiss Parameters for Axial-symmetric Solenoidal Focusing

Canonical variables

Canonical momenta:

$$p_x = x' - \frac{R}{2} y,$$

$$p_y = y' + \frac{R}{2} x.$$

Relation between geometrical and canonical variables

$$\hat{\mathbf{x}} = \mathbf{R}\mathbf{x} \quad ,$$

where

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} \quad , \quad \mathbf{x} \equiv \begin{bmatrix} x \\ q_x \\ y \\ q_y \end{bmatrix} \quad , \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

$R = eB_s / Pc$ - R is proportional to longitudinal magnetic field

Here and below we put a cap above transfer matrices and vectors related to the canonical variables.

- One can write for a single-particle phase-space trajectory along the beam orbit[1]

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\sqrt{e_1}\hat{\mathbf{v}}_1(s)e^{-i(y_1+m_1(s))} + \sqrt{e_2}\hat{\mathbf{v}}_2(s)e^{-i(y_2+m_2(s))}\right) \quad ,$$

  - vectors $\hat{\mathbf{v}}_1(s)$ and $\hat{\mathbf{v}}_2(s)$ are the eigen-vectors at coordinate s
  - y_1 and y_2 are the initial phases of betatron motion
  - $\sqrt{e_1} \equiv I_1$ and $\sqrt{e_2} \equiv I_2$ are the actions
- One can rewrite the above equations in the following form

$$\boxed{\hat{\mathbf{x}}(s) = \hat{\mathbf{V}}(s)\mathbf{a}(s)}$$

where

$$\hat{\mathbf{V}}(s) = \left[\hat{\mathbf{v}}_1'(s), -\hat{\mathbf{v}}_1''(s), \hat{\mathbf{v}}_2'(s), -\hat{\mathbf{v}}_2''(s)\right], \qquad \mathbf{a}(s) = \begin{bmatrix} \sqrt{e_1}\cos(y_1 + m_1(s)) \\ \sqrt{e_1}\sin(y_1 + m_1(s)) \\ \sqrt{e_2}\cos(y_2 + m_2(s)) \\ \sqrt{e_2}\sin(y_2 + m_2(s)) \end{bmatrix}$$

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[1] **BETATRON MOTION WITH COUPLING OF HORIZONTAL AND VERTICAL DEGREES OF FREEDOM**, V. Lebedev, S. Bogacz, **JLab-TN-00-022 ,** <http://tnweb.jlab.org/tn/2000/>

- **Eigen vector parameterization**
  - **Edwards and Teng**

$$\hat{\mathbf{v}}_1(s) = \mathbf{R}^{-1}\begin{bmatrix} \sqrt{b_1} \\ -\dfrac{i+a_1}{\sqrt{b_1}} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{v}}_2(s) = \mathbf{R}^{-1}\begin{bmatrix} 0 \\ 0 \\ \sqrt{b_2} \\ -\dfrac{i+a_2}{\sqrt{b_2}} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \cos f & 0 & -d\sin f & b\sin f \\ 0 & \cos f & c\sin f & -a\sin f \\ a\sin f & b\sin f & \cos f & 0 \\ c\sin f & d\sin f & 0 & \cos f \end{bmatrix}$$

  - **Ripken, *et al*.**

$$\hat{\mathbf{v}}_1(s) = \begin{bmatrix} \sqrt{b_{1x}(s)} \\ -\dfrac{i(1-u(s))+a_{1x}(s)}{\sqrt{b_{1x}(s)}} \\ \sqrt{b_{1y}(s)}e^{in_1(s)} \\ -\dfrac{iu(s)+a_{1y}(s)}{\sqrt{b_{1y}(s)}}e^{in_1(s)} \end{bmatrix}, \quad \hat{\mathbf{v}}_2(s) = \begin{bmatrix} \sqrt{b_{2x}(s)}e^{in_2(s)} \\ -\dfrac{iu(s)+a_{2x}(s)}{\sqrt{b_{2x}(s)}}e^{in_2(s)} \\ \sqrt{b_{2y}(s)} \\ -\dfrac{i(1-u(s))+a_{2y}(s)}{\sqrt{b_{2y}(s)}} \end{bmatrix},$$

In the case of axial symmetric focusing the eigen-vectors are (for simplicity we also skip dependence on s):

- **Edwards and Teng**

$$\hat{\mathbf{v}}_1 = \mathbf{R}^{-1}\begin{bmatrix} \sqrt{\tilde{b}} \\ -\dfrac{i+\tilde{a}}{\sqrt{\tilde{b}}} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \mathbf{R}^{-1}\begin{bmatrix} 0 \\ 0 \\ \sqrt{\tilde{b}_2} \\ -\dfrac{i+\tilde{a}_2}{\sqrt{\tilde{b}_2}} \end{bmatrix}, \quad \mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & \tilde{a} & \tilde{b} \\ 0 & 1 & -\dfrac{1+\tilde{a}^2}{\tilde{b}} & -\tilde{a} \\ \tilde{a} & \tilde{b} & 1 & 0 \\ -\dfrac{1+\tilde{a}^2}{\tilde{b}} & -\tilde{a} & 0 & 1 \end{bmatrix}$$

- **Ripken, *et al*.** $\tilde{a} = 2a \quad , \quad \tilde{b} = 2b$

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{b} \\ -\dfrac{i+2a}{2\sqrt{b}} \\ i\sqrt{b} \\ -i\dfrac{i+2a}{2\sqrt{b}} \end{bmatrix} \quad , \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} i\sqrt{b} \\ -i\dfrac{i+2a}{2\sqrt{b}} \\ \sqrt{b} \\ -\dfrac{i+2a}{2\sqrt{b}} \end{bmatrix}$$

Parameterization for Matrix **V**

$$\hat{\mathbf{V}} = \begin{bmatrix} \sqrt{b} & 0 & 0 & -\sqrt{b} \\ -\dfrac{a}{\sqrt{b}} & \dfrac{1}{2\sqrt{b}} & \dfrac{1}{2\sqrt{b}} & \dfrac{a}{\sqrt{b}} \\ 0 & -\sqrt{b} & \sqrt{b} & 0 \\ \dfrac{1}{2\sqrt{b}} & \dfrac{a}{\sqrt{b}} & -\dfrac{a}{\sqrt{b}} & \dfrac{1}{2\sqrt{b}} \end{bmatrix}$$

where we used that $u = 1/2$, $\nu_1 = \nu_2 = \pi/2$

2. Cooling Description

- One can write for ionization cooling due to energy loss on a thin absorber

$$\Delta q_{\perp} = -q_{\perp}\frac{\Delta p}{p} \equiv -q_{\perp} d$$

where we presume that the longitudinal energy is restored by acceleration. In matrix form that yields

$$\hat{\mathbf{x}}_{out} = \mathbf{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-d \end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{x}}_{in}$$

or for amplitudes

$$\hat{\mathbf{V}}\mathbf{a}_{out} = \mathbf{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-d \end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{V}}\mathbf{a}_{in} \Rightarrow \quad \mathbf{a}_{out} = \hat{\mathbf{V}}^{-1}\mathbf{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-d \end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{V}}\mathbf{a}_{in}$$

- Performing calculations one obtains

$$\mathbf{a}_{out} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - d \begin{bmatrix} \frac{1-Rb}{2} & a & -a & \frac{1+Rb}{2} \\ -a & \frac{1-Rb}{2} & \frac{1+Rb}{2} & a \\ -a & \frac{1-Rb}{2} & \frac{1+Rb}{2} & a \\ \frac{1-Rb}{2} & a & -a & \frac{1+Rb}{2} \end{bmatrix} \right\} \mathbf{a}_{in}$$

- Calculating values of the actions after cooling one obtains

$$e_1' \equiv a_{out_1}{}^2 + a_{out_2}{}^2 = e_1\left[1-(1-bR)d\right] + \sqrt{e_1 e_2}\left[2a\cos f - (1+bR)\sin f\right]d + O(d^2)$$

$$e_2' \equiv a_{out_3}{}^2 + a_{out_4}{}^2 = e_2\left[1-(1+bR)d\right] + \sqrt{e_1 e_2}\left[2a\cos f - (1-bR)\sin f\right]d + O(d^2)$$

where: $f = m_1 + m_2 + y_1 + y_2$

- Two descriptions of the cooling
  - After each absorber we compute
    - new 4D phase space
    - new partial emittances
    - new beta-functions
  - Or we compute everything relative to unperturbed beta-functions
    - More productive approach although partial emittances (actions) depend on betatron phases

    If cooling effect of one absorber is sufficiently small one can perform averaging over betatron phases. That yields

$$\Delta e_1 \approx -e_1(1-bR)d$$
$$\Delta e_2 \approx -e_2(1+bR)d$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{1}{e_1}\frac{de_1}{ds} &\approx -\frac{1-bR}{p_0}\frac{dp}{ds} \\ \frac{1}{e_2}\frac{de_2}{ds} &\approx -\frac{1+bR}{p_0}\frac{dp}{ds} \end{aligned}}$$

3. Canonical Angular Momentum and Second Order Moments

- Canonical momentum of a single particle

$$\mathbf{M} = xp_y - yp_x = \hat{\mathbf{x}}^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}} = \left(\hat{\mathbf{V}}\mathbf{a}\right)^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{V}}\mathbf{a} = \frac{e_1 - e_2}{2}$$

- Second order moments of the Gaussian distribution

(note that for a single particle - $e_{rms} = e/2$ and we use rms emittances below)

$$\left\langle x^2 \right\rangle = \left\langle y^2 \right\rangle = b\left(e_1 + e_2\right) \quad ,$$

$$\left\langle xp_x \right\rangle = \left\langle yp_y \right\rangle = -a\left(e_1 + e_2\right) \quad ,$$

$$\left\langle p_x{}^2 \right\rangle = \left\langle p_y{}^2 \right\rangle = \frac{1 + 4a^2}{4b}\left(e_1 + e_2\right) \quad ,$$

$$\left\langle xp_y \right\rangle = -\left\langle yp_x \right\rangle = \frac{e_1 - e_2}{2} \quad , \quad \Rightarrow \quad \left\langle M \right\rangle = e_1 - e_2$$

$$\left\langle xy \right\rangle = \left\langle p_x p_y \right\rangle = 0$$

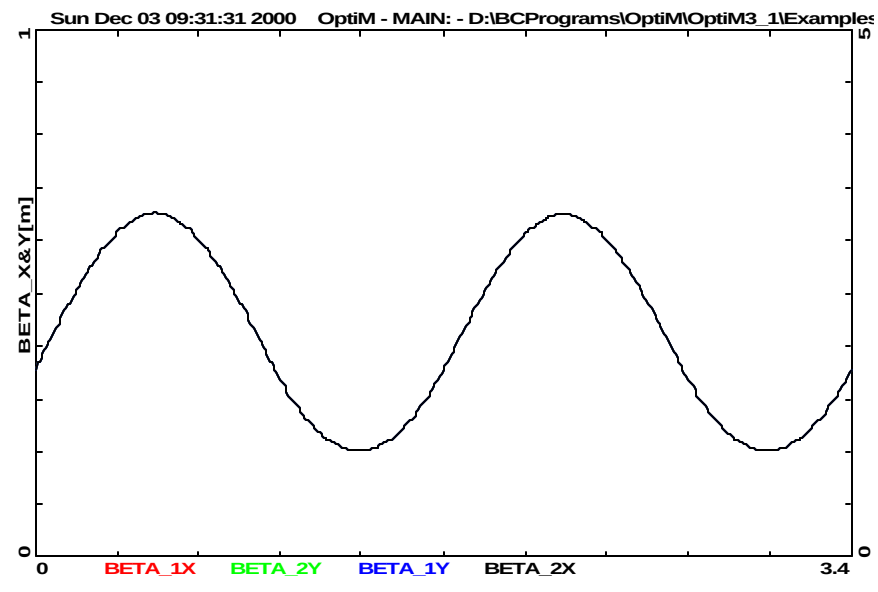
4. Beta-function for Particle Motion with Axial-symmetric Solenoidal Focusing

Equation for the square root of the beta-function is similar to the equation for Floque-function in the case of uncoupled motion:

$$\frac{d^2\sqrt{b}}{ds^2} + \frac{R^2}{4}\sqrt{b} - \frac{1}{4\left(\sqrt{b}\right)^3} = 0 \quad .$$

The alpha-function is determined by the standard recipe:

$$a = -\frac{1}{2}\frac{db}{ds}$$



Beta-function for two counter-wound solenoids