Fermilab

Stochastic Cooling with Schottky Band Overlap

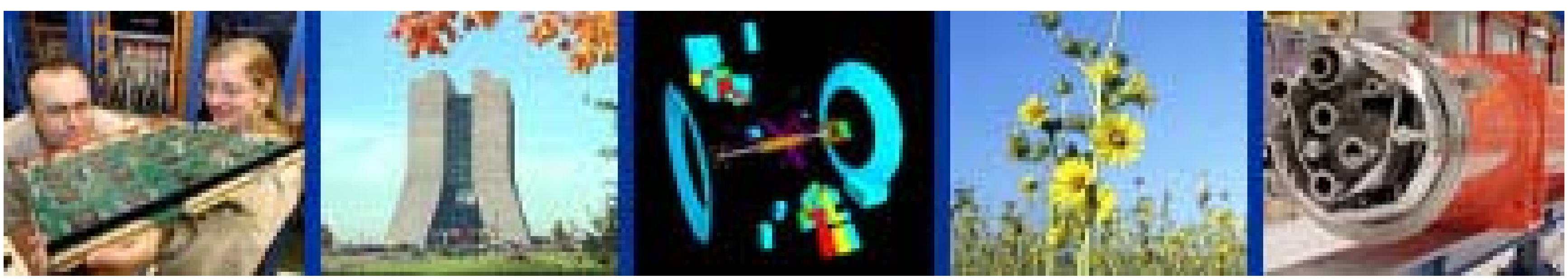
Valeri Lebedev

FNAL

COOL'05

Sep. 18-23, 2005

Galena, IL, USA



Motivation

1. Practical cooling systems are designed and built to avoid Schottky band overlap which compromises performance (safety factor ~1.5 is usually used)
2. Operating cooling systems are frequently used in a regime when bands are close to an overlap or slightly overlapped.
   a) In this case, the band overlap need to be taken into account if detailed description of the cooling is required.
   b) Quantitative description should be helpful for cooling optimization
3. Presently, all stochastic cooling calculations at FNAL are done neglecting effect of band overlap
   a) It needs to be corrected
   b) Taking band overlap into account should help to bring together the model calculations and the experimental observations (10-20% effect)

4. Some oddities in earlier papers

For example following expressions are proposed in Ref. [1]

a) Filter method

$$\varepsilon_{\mp|l|} = 1 + \frac{\pi N}{|l||\eta|\omega}\frac{\beta^2 E}{\omega} G_{\mp|l|}(\omega) \int_{\sigma\to 0_+} \frac{d\psi}{dx'} \frac{dx'}{\sigma \pm i(x-x')}$$

b) Palmer method

$$\varepsilon_{\mp|l|} = 1 + \frac{\pi N}{|l||\eta|\omega}\frac{\beta^2 E}{\omega} \int_{\sigma\to 0_+} \frac{d\psi}{dx'} \frac{G_{\mp|l|}(x')dx'}{\sigma \pm i(x-x')}$$

G(x) is linear without band overlap for both cooling methods
$\Rightarrow$ These equations are identical in the range of their applicability. While authors point out that the difference is important.

[1] J. Bisognano and C. Leemann, "Stochastic Cooling" in *1981 Summer School on high Energy Particle Accelerators,* edited by R. A. Carrigan et al., AIP Conference Proceedings 87, American Institute of Physics, Melville, NY, 1982, pp. 584-655.

Theorem: For linear $G_l(x)$ ($G_l(x) = G_0 + G'x$) the following equality is justified:

$$G_l(x) \int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{dx'}{\sigma \pm i(x - x')} = \int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{G_l(x')dx'}{\sigma \pm i(x - x')}$$

Proof:

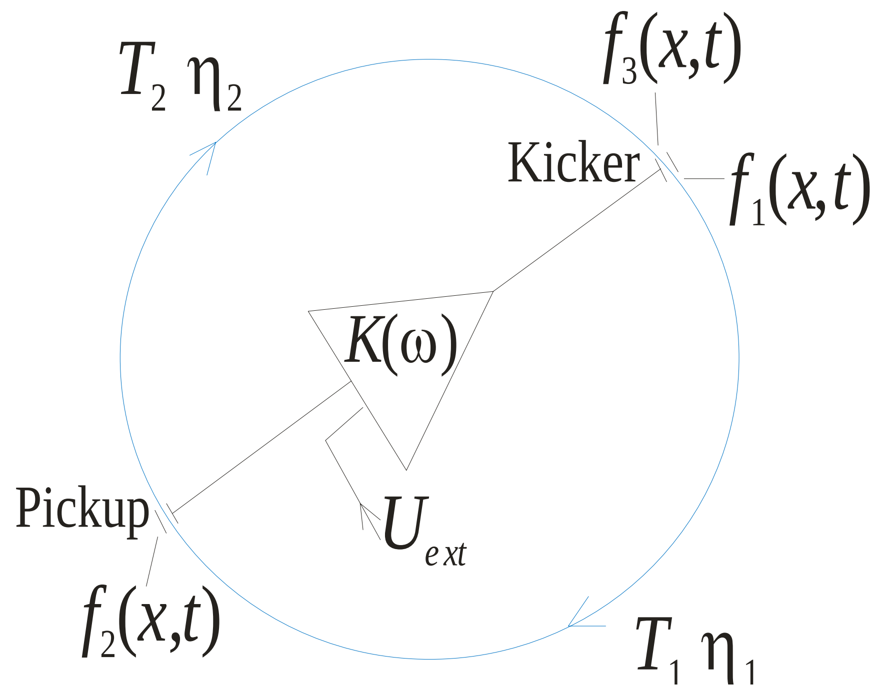
$$\int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{G_l(x')dx'}{\sigma \pm i(x - x')} - G_l(x) \int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{dx'}{\sigma \pm i(x - x')} =$$

$$\int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{G_l(x') - G_l(x)}{\sigma \pm i(x - x')} dx' = \int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{(G_0 + G'x') - (G_0 + G'x)}{\sigma \pm i(x - x')} dx' =$$

$$G' \int_{\sigma \to 0_+} \frac{d\psi}{dx'} \frac{x' - x}{\sigma \pm i(x - x')} dx' = -G' \int_{\sigma \to 0_+} \frac{d\psi}{dx'} dx' = 0$$

Beam Dielectric Permeability for Longitudinal Cooling

$$\begin{cases} f_2(x,t) = f_1(x, t - T_1(1+\eta_1 x_0)) \quad , \\ f_3(x,t) = f_2(x, t - T_2(1+\eta_2 x)) \quad , \\ f_1(x,t) = f_3(x - \delta p(t)/p_0, t)) \quad . \end{cases}$$



Schematic of longitudinal cooling system

- $x = (p - p_0)/p_0$ is the relative momentum deviation
- T_1, T_2 and $T_0 = T_1 + T_2$ are the kicker-to-pickup, pickup-to-kicker and revolution times for the reference particle
- $\eta = \alpha - 1/\gamma^2$ is the slip factor, η_1 and η_2 are the partial kicker-to-pickup and pickup-to-kicker slip factors, $\eta T_0 = \eta_1 T_1 + \eta_2 T_2$
- $\delta p(t)$ is the particle momentum change by the kicker

- Linearizing, performing Fourier transform and solving obtained equations relative $f_{2\omega}(x)$ yields:

$$\tilde{f}_{2\omega}(x)\exp(i\omega T_1(1+\eta_1 x)) = \tilde{f}_{2\omega}(x)\exp(-i\omega T_2(1+\eta_2 x)) - \frac{df_0(x)}{dx}\frac{\delta p_\omega}{p_0} \quad (1)$$

- We consider the case when the momentum kick is determined by the sum of amplified pickup signal and an external harmonic perturbation so that:

$$\delta p_\omega / p_0 = \int dx\, \tilde{f}_{2\omega}(x) G(x,\omega) e^{-i\omega T_2}\left[1 - A(\omega)e^{-i\omega T_0}\right] + \Delta p_{ext} / p_0 \qquad (2)$$

where: $G(x,\omega) = \dfrac{eI_0 Z_p(x,\omega) Z_k(\omega)}{\gamma\beta^2 mc^2 Z_{ampl}} K(\omega)$

I_0 – beam current; $Z_{ampl} = 50\,\Omega$

$e^{-i\omega T_2}$ takes into account the pickup-to-kicker signal delay, equal to the travel time for reference particle

  - Palmer cooling: $A(\omega)=0$
  - Filter cooling: $A(\omega)=1$ (notch filter is on), $G(x,\omega) \Rightarrow G(\omega)$

- Substituting (2) to (1) and solving obtained equation results:

$$S_\omega \equiv \int dx' \tilde{f}_{2\omega}(x') G(x',\omega) = -\frac{1}{\varepsilon(\omega)} \frac{\Delta p_{ext}}{p_0} \int_{\delta \to 0_+} \frac{df_0(x)}{dx} \frac{G(x',\omega) e^{i\omega T_2 (1+\eta_2 x)}}{e^{i\omega T (1+\eta x)} - (1-\delta)} dx$$

where $\varepsilon(\omega)$ is the beam dielectric permeability

$$\boxed{\varepsilon(\omega) = 1 + \left(1 - A(\omega) e^{-i\omega T_0}\right) \int_{\delta \to 0_+} \frac{df_0(x)}{dx} \frac{G(x,\omega) e^{i\omega T_2 \eta_2 x}}{e^{i\omega T_0 (1+\eta x)} - (1-\delta)} dx}$$

- Far away from Schottky band overlap the exponent in the denominator can be expended in vicinity of revolution harmonic, $\omega = n\omega_0 + \delta\omega$, $\omega_0 = 2\pi / T_0$ and we arrive to the standard formula for the dielectric permeability

$$\varepsilon_n(\delta\omega) = 1 + \left(1 - A(\omega) e^{-i\omega T_0}\right) \int_{\delta \to 0_+} \frac{df_0(x)}{dx} \frac{G(x,\omega)}{2\pi i \left(\eta n x + \delta\omega / \omega_0 - i\delta\right)} dx$$

Beam Dielectric Permeability for Transverse Cooling

Beam dipole moment at each point is

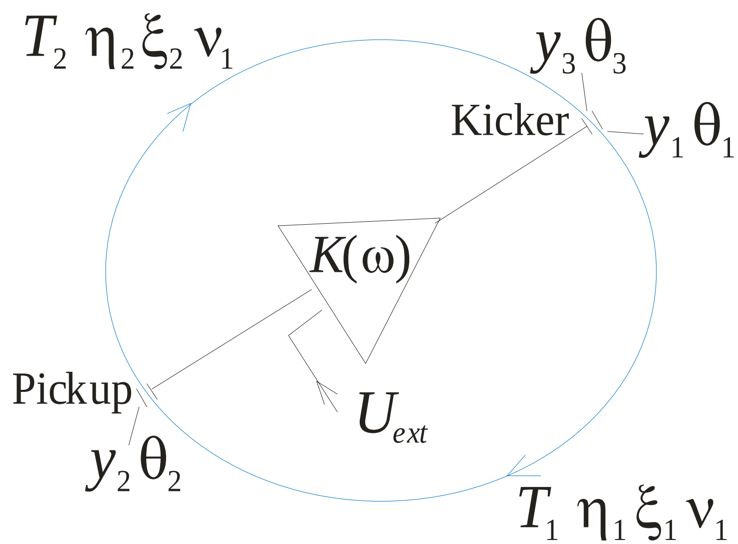
$$d_k(t) = \frac{I_0}{c\beta}\int y_k(x) f_0(x)dx \quad , \quad k = 1,2,3. \;.$$

Normalizing x and θ by the β-functions

$$\tilde{y}_k = y_k / \sqrt{\beta_k}$$

$$\tilde{\theta}_k = \theta_k\sqrt{\beta_k} - \alpha_k x_k / \sqrt{\beta_k}$$

we obtain equations relating the beam positions after and before the kicker:



Schematic of transverse cooling system.

$$\tilde{y}_3(x,t) = c(x)\tilde{y}_1(x, t - T_0(1+\eta x)) + s(x)\tilde{\theta}_1(x, t - T_0(1+\eta x)) \quad ,$$
$$\tilde{\theta}_3(x,t) = -s(x)\tilde{y}_1(x, t - T_0(1+\eta x)) + c(x)\tilde{\theta}_1(x, t - T_0(1+\eta x)) \quad . \qquad (3)$$

Here $c(x) = \cos(2\pi(\nu + \xi x))$, $s(x) = \sin(2\pi(\nu + \xi x))$, ν is the betatron tune, and ξ is the tune chromaticity.

- Passing the kicker changes y but does not change θ

$$\tilde{y}_1(x,t) = \tilde{y}_3(x,t) \quad , \qquad \tilde{\theta}_1(x,t) = \tilde{\theta}_3(x,t) + \delta\tilde{\theta}(t) \quad . \qquad (4)$$

- Performing Fourier transform in Eqs. (3) and (4) and solving them relative to $\tilde{y}_{1\omega}(x)$ and $\tilde{\theta}_{1\omega}(x)$ we obtain:

$$\tilde{\theta}_{1\omega}(x) = -\frac{\left(c(x) - \exp\left(i\omega T_0(1+\eta x)\right)\right)\exp\left(i\omega T_0(1+\eta x)\right)}{\exp\left(2i\omega T_0(1+\eta x)\right) - 2c(x)\exp\left(i\omega T_0(1+\eta x)\right) + 1}\,\delta\tilde{\theta}_\omega \quad ,$$

$$\tilde{y}_{1\omega}(x) = \frac{s(x)\exp\left(i\omega T_0(1+\eta x)\right)}{\exp\left(2i\omega T_0(1+\eta x)\right) - 2c(x)\exp\left(i\omega T_0(1+\eta x)\right) + 1}\,\delta\tilde{\theta}_\omega \quad .$$

- That yields displacement in the pickup

$$\tilde{y}_{2\omega}(x) = \frac{\left(s_2(x) + s_1(x)e^{i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1}\,\delta\tilde{\theta}_\omega \qquad (5)$$

Here $c_{1,2}(x) = \cos\left(2\pi(\nu_{1,2} + \xi_{1,2}x)\right)$, $s_{1,2}(x) = \sin\left(2\pi(\nu_{1,2} + \xi_{1,2}x)\right)$,
$2\pi\nu_1$ and $2\pi\nu_2$ are the betatron phase advances between pickup and kicker so that $\nu_1 + \nu_2 = \nu$,
ξ_1 and ξ_2 are the partial tune chromaticities, $\xi_1 + \xi_2 = \xi$.

- The beam kick is determined by the sum of amplified pickup signal and an external harmonic perturbation

$$\delta\tilde{\theta}_\omega = \int dx\, f_0(x)\tilde{y}_{2\omega}(x)G_\perp(\omega)e^{-i\omega T_2} + \Delta\tilde{\theta}_{ext} \qquad (6)$$

where the gain is $G_\perp(\omega) = \dfrac{eI_0 Z_{p\perp}(\omega)Z_{k\perp}(\omega)}{\gamma\beta^2 mc^2 Z_{ampl}}\sqrt{\beta_p\beta_k}K(\omega)$

β_p and β_k are the beta-functions in the pickup and kicker.

- Combining Eqs. (5) and (6) we obtain:

$$\tilde{y}_{2\omega}(x) = \frac{\left(s_2(x) + s_1(x)e^{2i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1}\left(G_\perp(\omega)e^{-i\omega T_2}\int dx\, f_0(x)\tilde{y}_{2\omega}(x) + \Delta\tilde{\theta}_{ext}\right)$$

- The solution is similar to the solution for longitudinal cooling. It yields the average beam displacement in the pickup:

$$\overline{\tilde{y}_{2\omega}} \equiv \int dx\, f_0(x)\tilde{y}_{2\omega}(x) = \frac{\Delta\tilde{\theta}_{ext}}{\varepsilon_\perp(\omega)}\int dx\, f_0(x)\frac{\left(s_2(x) + s_1(x)e^{i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1}$$

where the beam permeability is:

$$\boxed{\varepsilon_\perp(\omega) = 1 - \frac{G_\perp(\omega)}{2}\int\limits_{\delta\to 0_+}\frac{\left[e^{-i\omega T_0(1+\eta x)}\sin(2\pi(\nu_2 + \xi_2 x)) + \sin(2\pi(\nu_1 + \xi_1 x))\right]e^{i\omega T_2\eta_2 x}}{\cos(\omega T_0(1+\eta x)) - \cos(2\pi(\nu + \xi x)) + i\delta\sin(\omega T_0(1+\eta x))}f_0(x)dx}$$

Fokker-Planck Equations

Longitudinal cooling

- Evolution of the distribution function is described by:

$$\boxed{\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}\left(F(x)f\right)=\frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right)}$$

- The drag force is created by the particle self-interaction and therefore is not directly affected by the band overlap; but it is affected by screening of the particle signal

$$\boxed{F(x)\equiv\frac{dx}{dt}=\frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{G_1(x,\omega_n)}{\varepsilon(\omega_n)}\left(1-A(\omega_n)e^{-i\omega_n T_0}\right)e^{i\omega_n T_2\eta_2 x}}$$

Here $G_1(x,\omega)=G(x,\omega)/N$ is the single particle gain,

$\omega_n=\omega_0(1-\eta x)n$,

N is the particle number in the beam,

$\varepsilon(\omega_n)$ in the denominator takes into account screening effect

- The diffusion is created by noise in the kicker voltage:

$$D(x) = \frac{2\pi e^2}{T_0{}^2\left(\gamma\beta^2 mc^2\right)^2}\sum_{n=-\infty}^{\infty} P_U(\omega_n) \qquad (7)$$

where $P_U(\omega)$ is the spectral density of kicker voltage consisting of two contributions. The first one is related to the noise of the electronics, P_{Unoise}, and the second one is related to the particle shot noise.

- The beam current shot noise for non-interacting particles is:

$$P_I(\omega) = \frac{e^2 N}{2\pi T_0}\sum_{k=-\infty}^{\infty}\frac{1}{|k\eta|} f\left(\frac{\omega - k\omega_0}{\eta k\omega_0}\right) \quad . \qquad (8)$$

- Combining Eqs. (27) and (28) and simplifying one obtains:

$$\boxed{D(x) = \sum_{n=-\infty}^{\infty}\frac{1}{|\varepsilon(\omega_n)|^2}\left[\frac{2\pi e^2 P_{Unoise}(\omega_n)}{T_0{}^2\left(\gamma\beta^2 mc^2\right)^2} + \frac{N}{T_0}\left|G_1(x,\omega_n)\left(1 - A(\omega_n)e^{-i\omega_n T_0}\right)\right|^2 \cdot \sum_{k=-\infty}^{\infty}\frac{1}{|k\eta|} f\left(\frac{(1+\eta x)n - k}{\eta k}\right)\right]}$$

Transverse cooling

- Natural variables for transverse cooling description are the action-phase variables (I, ψ): $I=\left(\beta_y\theta^2+2\alpha_y y\theta+\left(1+\alpha_y{}^2\right)y^2/\beta_y\right)/2$, where β_y and α_y are the beta- and alpha-functions of the ring.
- We assume that there is no *x-y* coupling in the lattice, and the cooling is linear in betatron amplitude. Then the beam distribution function is described by the following equation:

$$\boxed{\frac{\partial f_\perp}{\partial t}+\lambda_\perp(x)\frac{\partial}{\partial I}\left(If_\perp\right)=D_\perp(x)\frac{\partial}{\partial I}\left(I\frac{\partial f_\perp}{\partial I}\right)}$$

Here $f_\perp\equiv f_\perp(x,I,t)$ is the distribution function normalized so that $\int f_\perp(x,I,t)dI=f_0(x)$ and $\int f_0(x)dx=1$,

$\lambda_\perp(x)$ is the cooling decrement

$D_\perp(x)$ is the diffusion coefficient.

Because of system linearity $\lambda_\perp(x)$ and $D_\perp(x)$ do not depend on I.

- Similar to the longitudinal cooling the transverse cooling is created by the particle self-interaction and therefore is not directly affected by the band overlap. But it is still affected by particle screening of signal by other particles. The result is:

$$\boxed{\lambda_\perp(x) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\mathrm{Re}\left(i\frac{G_{\perp 1}(\omega_n)}{\varepsilon_\perp(\omega_n)} e^{i\omega_n T_2\eta_2 x - 2\pi i \nu_2} \right) , \; \omega_n = \omega_0\left(n(1-\eta x) - (\nu + \xi x)\right)}$$

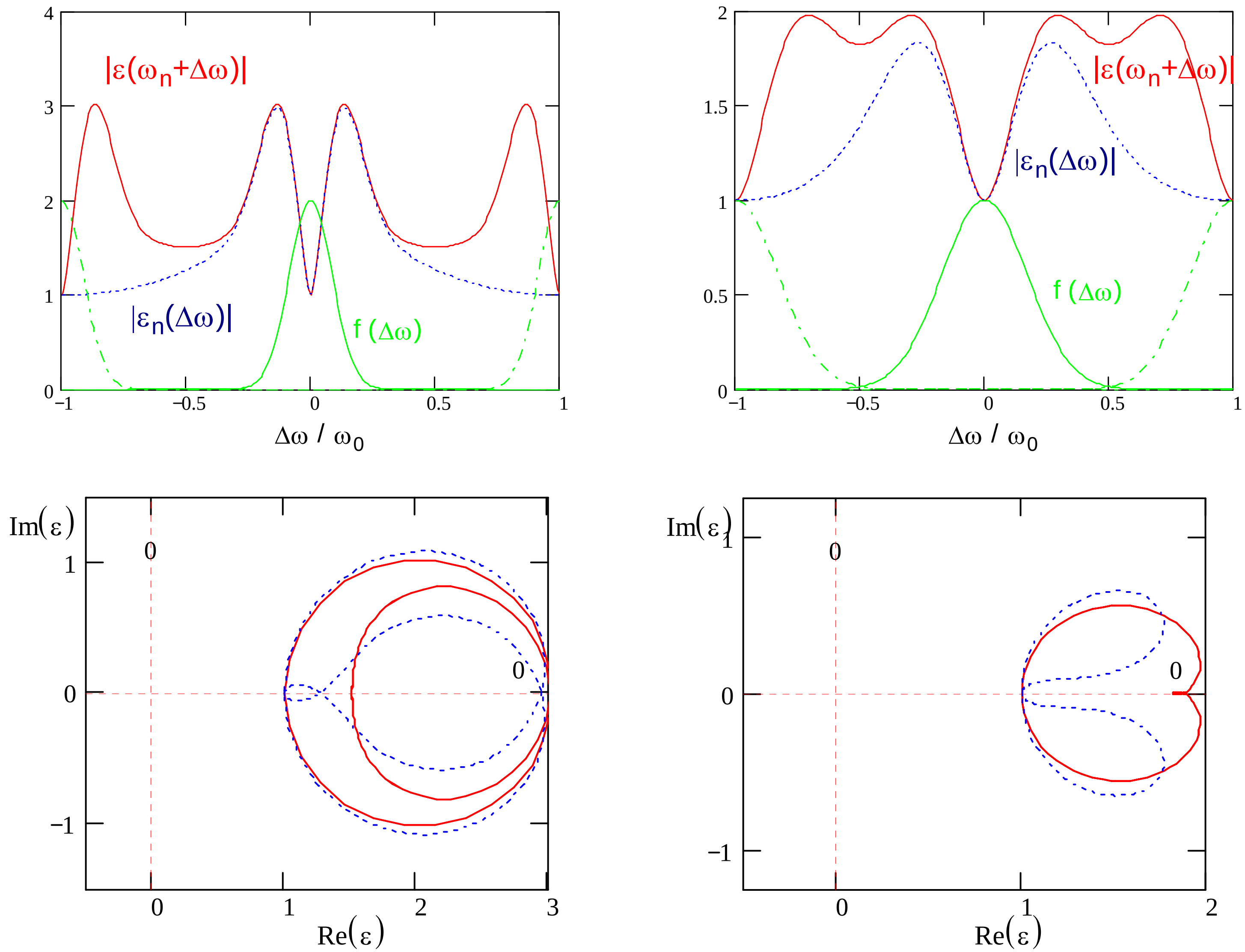
where $G_{\perp 1}(\omega) = G_\perp(\omega)/N$ is the single particle gain.

- The diffusion is created by kicker noise and is combined from own noise of power amplifier, $P_{\perp U}(\omega)$, and the particle shot noise suppressed by particle interaction:

$$\boxed{D_\perp(x) = \sum_{n=-\infty}^{\infty}\frac{1}{\left|\varepsilon_\perp(\omega_n)\right|^2}\left(\frac{\pi\beta_k}{2T_0^{\;2}}\left(\frac{e\left|Z_{k\perp}(\omega_n)\right|}{mc^2\beta^2\gamma Z_{ampl}}\right)^2 P_{\perp U}(\omega_n) + \left|G_{\perp 1}(\omega_n)\right|^2 \frac{\langle I(x)\rangle N}{2T_0}\sum_{m=-\infty}^{\infty}\frac{f\left(\dfrac{n-m+(\xi+\eta m)x}{\xi+\eta m}\right)}{\left|\xi+\eta m\right|} \right)}$$

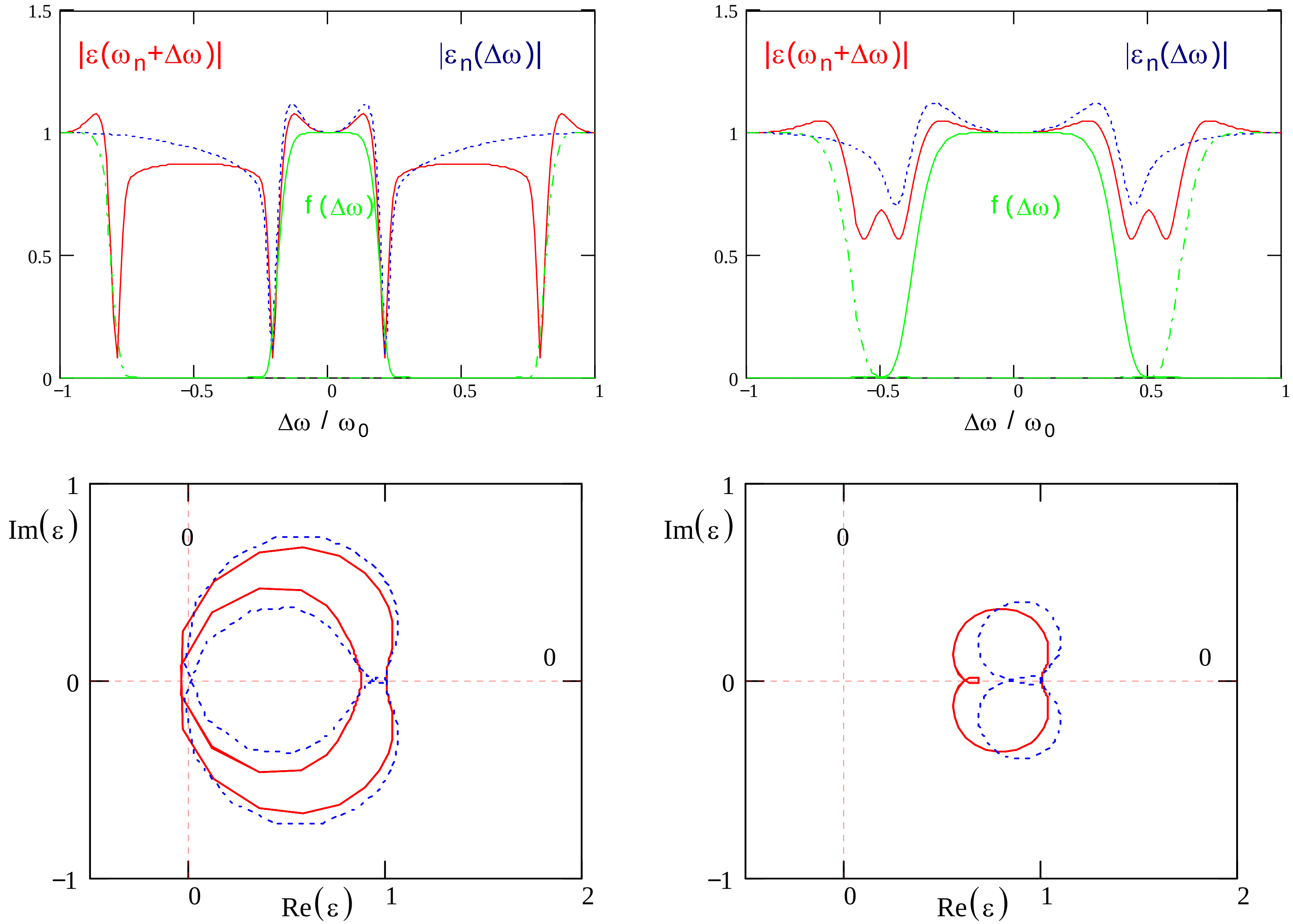
where $\langle I(x)\rangle = \int f_\perp(x,I,t) I dI$ is the average action for given momentum deviation x.

Example: Longitudinal cooling in Debuncher (Filter cooling)



σ_p=1.05 10^{-3} σ_p=2.1 10^{-3}

Gaussian distribution, $G(\omega)$ = 0.0023i , A=1, n = 12000, nf_0=7.078 GHz



$\Delta p/p$ = 2.2 10^{-3} $\Delta p/p$ = 4.4 10^{-3}

Flat distribution $f(x) \propto \exp\left(-(1/2)\,x^8/(\Delta p/p)^8\right)$, $G(\omega)$=0.000538i, A=1, n=12000, nf_0=7.078 GHz