

Single and Multiple Intrabeam Scattering in Hadron Colliders

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Talk outline

1. Introduction
 2. Growth rates for Gaussian beams
 3. IBS in non-linear longitudinal well
 4. Comparison with experiment
- Conclusions

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Brightness Hadron
Beams**

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1. Introduction

Objective

- ◆ Steady growth of the luminosity for last 3 years
 - Present record luminosity $10^{32} \text{ cm}^{-2}\text{s}^{-1}$
 - Three fold luminosity increase during next 2-3 years
 - Major part should come from increase of p-bar production
 - but the luminosity growth has to be supported by Tevatron
- ◆ Luminosity evolution in Tevatron is driven by
 - Elastic and non-elastic scattering on the residual gas
 - Elastic and non-elastic scattering on counter-rotating beam
 - RF noise & transverse noise (magnetic field fluctuations, quad motion, etc.)
 - Intrabeam scattering
 - Beam-beam effects
- ◆ The major aim for the effort is to understand better the beam-beam effects and other possible limitations of the luminosity
- ◆ To make any practical conclusions accurate measurements are not less important than good theory

IBS

- ◆ Growth rates for Gaussian beams (Bjorken, Mtingwa, 1982)
 - X-Y coupling needs to be taken into account in the case of Tevatron
 - This approximation is good for many applications but is not self-consistent
 - the beam does not stay gaussian in the course of evolution
 - The evolution of beam parameters is determined by system of ordinary differential equations
- ◆ Longitudinal degree of freedom is different from transverse ones
 - Non-linear focusing
 - Finite RF bucket size
 - The distribution function evolution is determined by partial diff. equation
- ◆ Simultaneous treatment of single and multiple scattering are required in many practical applications
 - Contributions from single and multiple scattering cannot be separated at the store beginning in Tevatron
 - Beam-life time in Recycler with cooling cannot be accurately computed if only single scattering is taken into account
 - The distribution function evolution is determined by integro-differential equation (Boltzmann type)

2. Growth rates for Gaussian beams

- ◆ Landau collision integral

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial p_i} (F_i f) = \frac{1}{2} \frac{\partial}{\partial p_i} \left(D_{ij} \frac{\partial f}{\partial p_j} \right)$$

$$F_i(p) = -\frac{4\mathbf{p}ne^4 L_c}{m} \int f(p') \frac{u_i}{|\mathbf{u}|^3} d^3 p' \quad , \quad \mathbf{u} = \mathbf{v} - \mathbf{v}'$$

$$D_{ij} = 4\mathbf{p}ne^4 L_c \int f(p') \frac{u^2 \mathbf{d}_{ij} - u_i u_j}{|\mathbf{u}|^3} d^3 p'$$

- ◆ Integration with Gaussian distrib. in all 3 degrees of freedom yields

$$\frac{d}{dt} \frac{1}{v_x^2} = \frac{(2\mathbf{p})^{3/2} ne^4 L}{m^2 \sqrt{v_x^2 + v_y^2 + v_z^2}} \Psi \left(\sqrt{v_x^2}, \sqrt{v_y^2}, \sqrt{v_z^2} \right)$$

- Equations for other degrees of freedom are obtained by cyclic substitution

Here:

$$\Psi(x, y, z) = \frac{\sqrt{x^2 + y^2 + z^2}}{\mathbf{p}} \int_0^\infty \frac{\sqrt{2} t^3 dt}{(x^2 + t^2)^{3/2} (y^2 + t^2)^{1/2} (z^2 + t^2)^{1/2}} \left(\frac{y^2 - x^2}{y^2 + t^2} + \frac{z^2 - x^2}{z^2 + t^2} \right)$$

- ◆ Energy conservation requires: $\Psi(x, y, z) + \Psi(y, z, x) + \Psi(z, x, y) = 0$

- ◆ The function $\Psi(x, y, z)$ does not depend on $x^2 + y^2 + z^2$ and therefore for further analysis we choose $x^2 + y^2 + z^2 = 1$

- ◆ The function $\Psi(x, y, z)$ is determined so that

$$\Psi(0, x, x) = 1 \quad \Psi(x, x, 0) = \Psi(x, 0, x) = -1/2 \quad \Psi(x, x, x) = 0 \quad (1)$$

- ◆ When two parameters coincide the integral can be computed

$$\Psi\left(x, \sqrt{\frac{1-x^2}{2}}, \sqrt{\frac{1-x^2}{2}}\right) = 2\hat{\Psi}(x) \quad \hat{\Psi}(x) = \frac{1}{\sqrt{2p}} \frac{1}{3x^2-1} \left(\frac{1}{\sqrt{2}} \frac{3x^2+1}{\sqrt{3x^2-1}} \ln\left(\frac{\sqrt{2}x - \sqrt{3x^2-1}}{\sqrt{2}x + \sqrt{3x^2-1}}\right) + 6x \right)$$

$$\Psi\left(\sqrt{\frac{1-z^2}{2}}, \sqrt{\frac{1-z^2}{2}}, z\right) = -\hat{\Psi}(z)$$

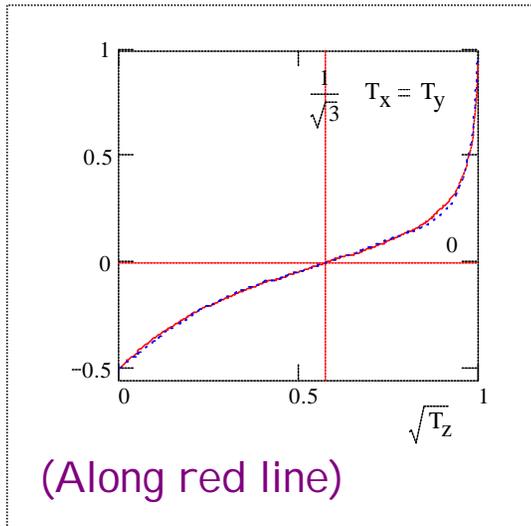
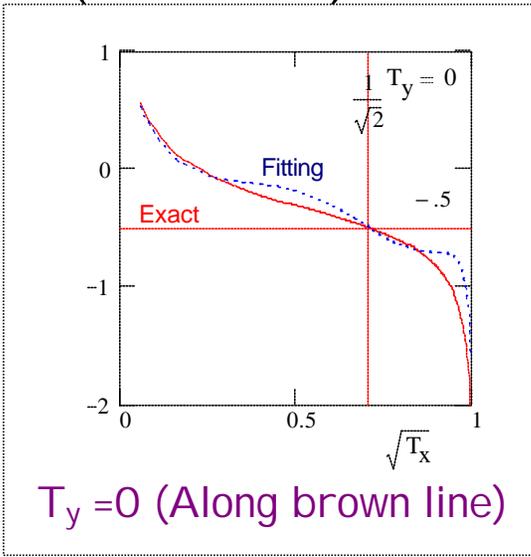
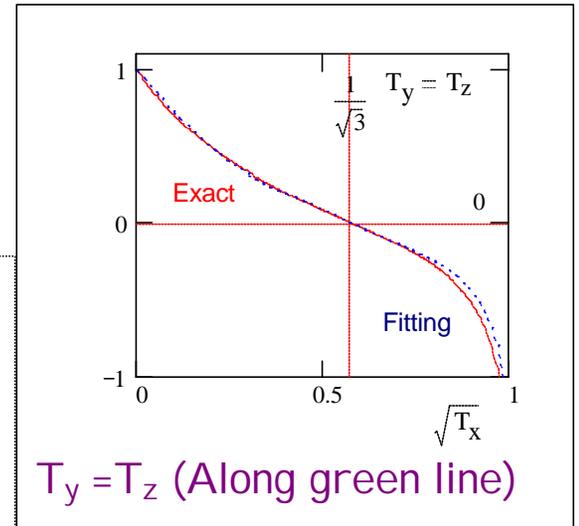
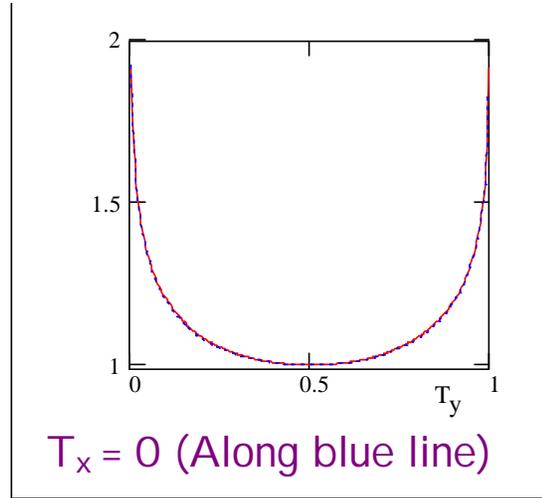
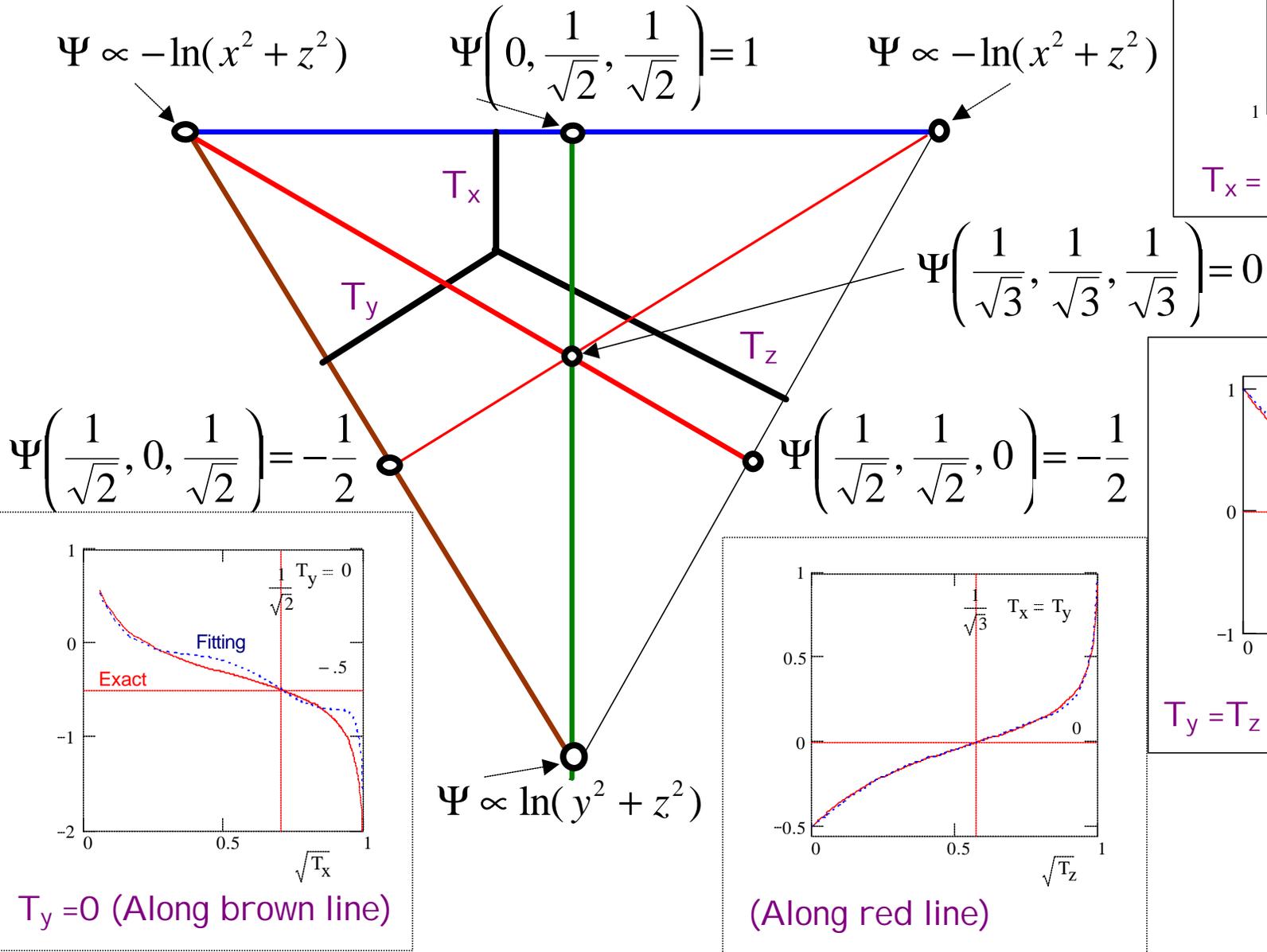
- ◆ For practical applications the function $\Psi(x, y, z)$ can be approximated as

$$\Psi(x, y, z) \approx \frac{\sqrt{2}}{p} \ln\left(\frac{y^2 + z^2}{\sqrt{x^2 + y^2} \sqrt{x^2 + z^2}}\right) + 0.688 \frac{(x-y)(x-z)}{\sqrt{x^2 + y^2} \sqrt{x^2 + z^2}} - 0.055 \left(\frac{y^2 - z^2}{r^2}\right)^2 -$$

$$7.47 \frac{x^6}{r^9} |y - z|^3 - 0.8 \frac{x}{r} \left(1 - 3 \frac{x^2}{r^2}\right)^3 \mathbf{J}\left(1 - 3 \frac{x^2}{r^2}\right), \quad \mathbf{J}(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

- This function has correct asymptotics, satisfy conditions (1), and coincide with exact expression within
- ~1% for $x=0$, ~10% for entire range of parameters

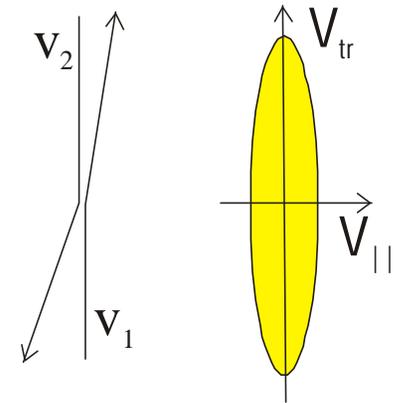
Dalitz plot for temperatures $T_x = x^2, T_y = y^2, T_z = z^2$,



Accelerator specific corrections

◆ General Recipe

- Find local density and velocity spreads and compute average temperature growth across the beam cross-section. Then average along beam orbit.
 - Take into account that axis of 3D ellipsoid of velocities not necessarily coincide with local coordinate frame axis
 - Take into account additional excitation of transverse degrees of freedom due to non-zero dispersions



◆ Details

- For Tevatron significant simplifications are due to
 - $v_{\parallel} \ll v_x, v_y$ in the beam frame
 - $g \gg Q_x, Q_y$

- After averaging over the bunch length and the cross-section

$$\frac{d}{dt} (\mathbf{s}_p^2) \equiv \left\langle \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s = \frac{1}{4\sqrt{2}} \frac{N r_p^2 c}{\mathbf{g}^3 \mathbf{b}^3} \left\langle \frac{1}{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_s} \frac{\Psi(\mathbf{s}_p / \mathbf{g}, \mathbf{q}_1, \mathbf{q}_2)}{\sqrt{\mathbf{q}_1^2 + \mathbf{q}_2^2 + (\mathbf{s}_p / \mathbf{g})^2}} L_C \right\rangle_s$$

- Here \mathbf{q}_1 and \mathbf{q}_2 are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and \mathbf{s}_1 and \mathbf{s}_2 are ellipse semi-axis in the x-y plane

➤ Uncoupled motion

$$\mathbf{s}_1 \equiv \mathbf{s}_x = \sqrt{\mathbf{e}_x \mathbf{b}_y + D^2 \mathbf{s}_p^2},$$

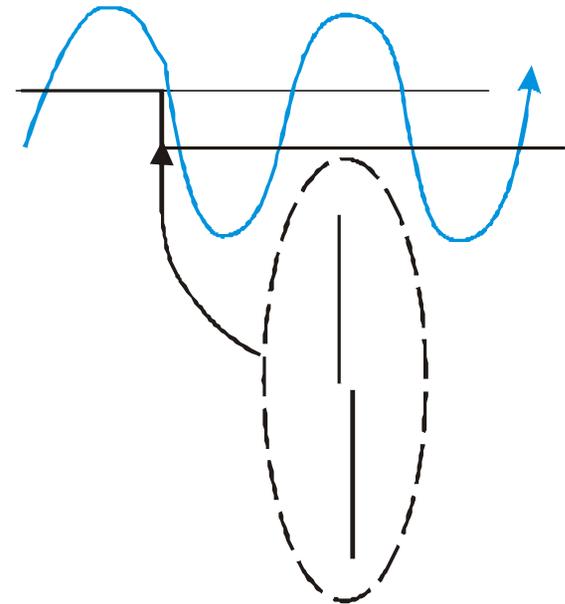
$$\mathbf{s}_2 \equiv \mathbf{s}_y = \sqrt{\mathbf{e}_y \mathbf{b}_y},$$

$$\mathbf{q}_1 \equiv \mathbf{q}_x = \sqrt{\frac{\mathbf{e}_x}{\mathbf{b}_x} \left(1 + \frac{(D' \mathbf{b}_x + \mathbf{a}_x D)^2 \mathbf{s}_p^2}{\mathbf{e}_x \mathbf{b}_x + D^2 \mathbf{s}_p^2} \right)},$$

$$\mathbf{q}_2 \equiv \mathbf{q}_y = \sqrt{\mathbf{e}_y / \mathbf{b}_y}$$

- Additional transverse emittance growth due to finite dispersion dominates emittance change due to “direct” scattering

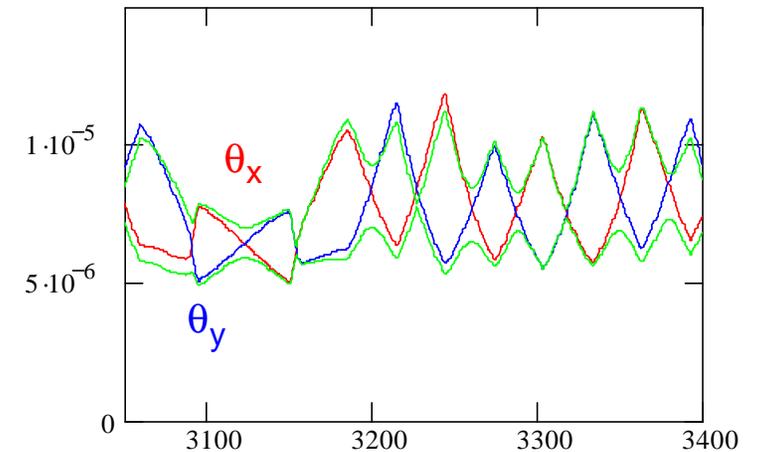
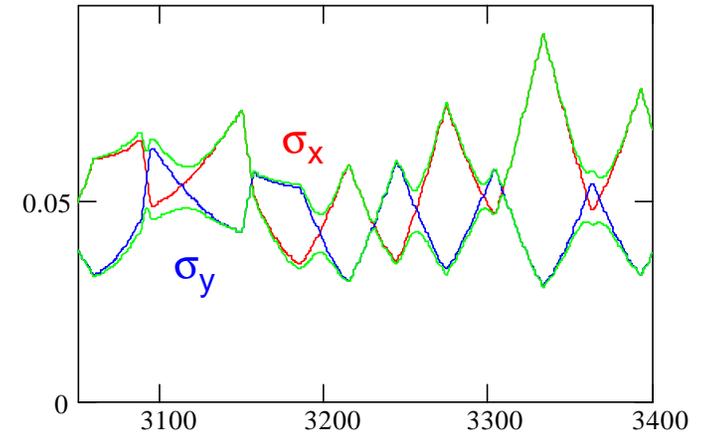
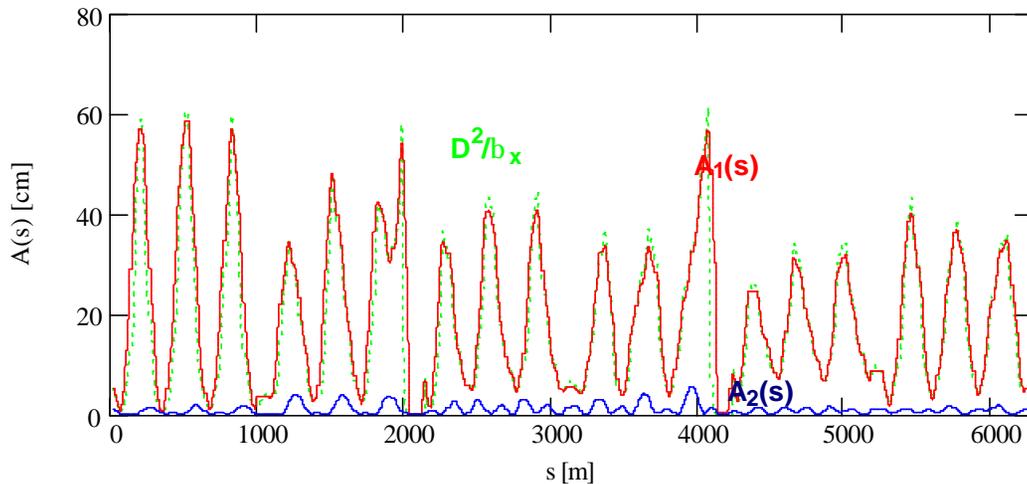
$$\frac{d\mathbf{e}_x}{dt} = \left\langle \frac{D^2 + (D' \mathbf{b}_x + \mathbf{a}_x D)^2}{\mathbf{b}_x} \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s$$



➤ X-Y coupled motion

- Measured Tevatron optics with coupling has been used in calculations!!!
- Coupling effects are sufficiently small
 - little corrections for density and angular spread
- Emittance growth related to mode 2 (y mode) is about 5% of mode 1 (x-mode)

$$\frac{d\mathbf{e}_1}{dt} = \left\langle A_1 \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s, \quad \frac{d\mathbf{e}_2}{dt} = \left\langle A_2 \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s$$



Top - Beam size projections and ellipsoid semi-axis
 Bottom - projections for angular spreads and ellipsoid semi-axis

- In measurements both modes contribute to the beam sizes

$$\mathbf{s}_x^2 = \mathbf{b}_{1x} \mathbf{e}_1 + \mathbf{b}_{2x} \mathbf{e}_2$$

$$\mathbf{s}_y^2 = \mathbf{b}_{1y} \mathbf{e}_1 + \mathbf{b}_{2y} \mathbf{e}_2$$

- That yields for observed emittance growth

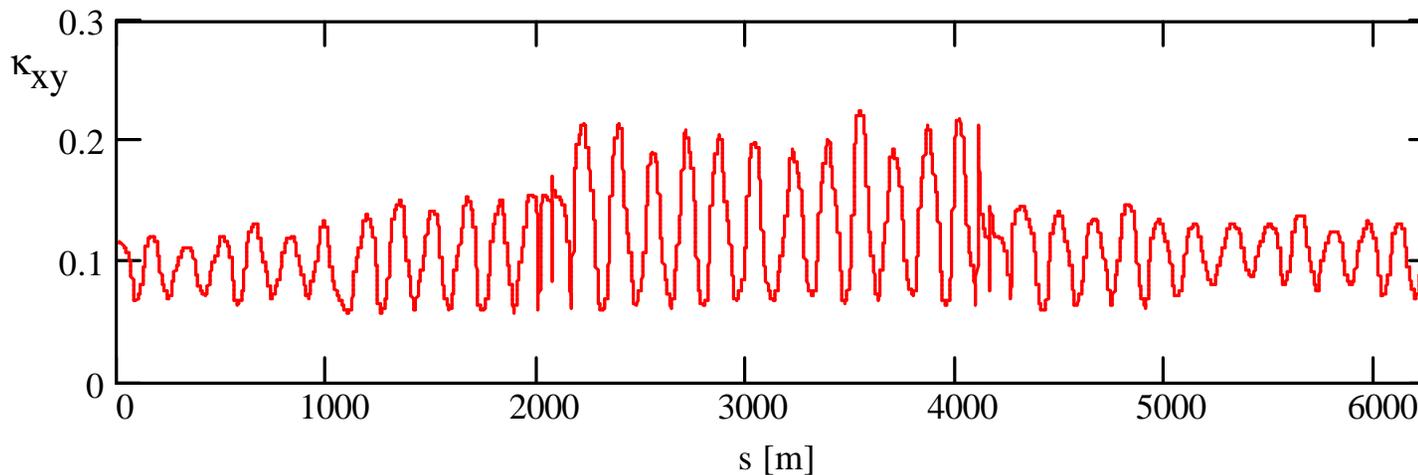
$$\frac{d\mathbf{e}_x}{dt} = \frac{d\mathbf{e}_1}{dt} + \frac{\mathbf{b}_{2x}}{\mathbf{b}_{1x}} \frac{d\mathbf{e}_2}{dt}$$

$$\frac{d\mathbf{e}_y}{dt} = \frac{\mathbf{b}_{1y}}{\mathbf{b}_{2y}} \frac{d\mathbf{e}_1}{dt} + \frac{d\mathbf{e}_2}{dt}$$

- That leads to an increase of observed coupling

- $\mathbf{k}_{xy} \approx 0.1$ at Synchrotron light emittance monitor

$$\mathbf{k}_{xy} \equiv \frac{d\mathbf{e}_y / dt}{d\mathbf{e}_x / dt + d\mathbf{e}_y / dt}$$



3. IBS in Non-linear Longitudinal Well

Diffusion equation

- ◆ In the case $v_{\parallel} \ll v_x, v_y$ the friction in Landau collision integral can be neglected

$$D(I) = \langle D(p) \rangle_{\text{period}}$$

- ◆ Diffusion equation

$$1D: \frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial p} \left(D(p) \frac{\partial f}{\partial p} \right) \Rightarrow 2D: \frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial I} \left(I \frac{D(I)}{w(I)} \frac{\partial f}{\partial I} \right)$$

- I is the action and w is the frequency for dimensionless Hamiltonian of synchrotron motion: $H = \frac{p^2}{2} + 2 \left(\sin \frac{j}{2} \right)^2$
- Diffusion coefficient depends on distribution, (I)

$$D(I) = 4L_c \tilde{A} \left(\oint n(\mathbf{j}) p d\mathbf{j} / \oint p d\mathbf{j} \right) \quad \tilde{A} = p^2 \sqrt{\frac{p}{2}} \frac{(a - 1/g^2) e^4 q^2}{e V_0 m_p c b g^2 C} \left\langle \frac{N}{\mathbf{s}_1 \mathbf{s}_2} \frac{\Psi(\mathbf{s}_p / \mathbf{g}, \mathbf{q}_1, \mathbf{q}_2)}{\sqrt{\mathbf{q}_1^2 + \mathbf{q}_2^2 + (\mathbf{s}_p / \mathbf{g})^2}} \right\rangle_s$$

Here: $n(\mathbf{j}) = \int f(I(p, \mathbf{j})) dp$, $\int_{-p}^p n(\mathbf{j}) d\mathbf{j} = 1$

a - momentum compaction,

q - harmonic number

V_0 - RF voltage,

C - ring circumference

Simultaneous treatment of single and multiple scattering

◆ Boltzmann type equation

➤ In the case $v_{\parallel} \ll v_{\perp}$ one can write for Coulomb scattering in long. direction

$$\frac{\partial f}{\partial t} = \left\langle \tilde{A} \int n(\mathbf{j}) \frac{f(p+q) - f(p)}{|q|^3} dq \right\rangle_{\text{period}} = \left\langle \tilde{A} \int n(\mathbf{j}) \frac{f(I') - f(I)}{|p-p'|^3} \mathbf{d}(\mathbf{j} - \mathbf{j}') dI' dy dy' \right\rangle_{\text{period}}$$

◆ After simplification we obtain

$$\frac{\partial f(I, t)}{\partial t} = \tilde{A} \int_0^{\infty} W(I, I') (f(I', t) - f(I, t)) dI'$$

$$W(I, I') = \frac{2\mathbf{w}\mathbf{w}'}{\mathbf{p}} \int_0^{\min(a, a')} \frac{d\mathbf{j}}{pp'} n(\mathbf{j}) \left[\frac{1}{|p-p'|^3} + \frac{1}{|p+p'|^3} \right] \xrightarrow{E' \geq E} \frac{\mathbf{w}\mathbf{w}'}{\mathbf{p}(E-E')^3} \left[(E-E') \int_0^a n(\mathbf{j}) \frac{dx}{p} + 2 \int_0^a n(x) p dx \right] .$$

$a \equiv a(I)$ is the motion amplitude

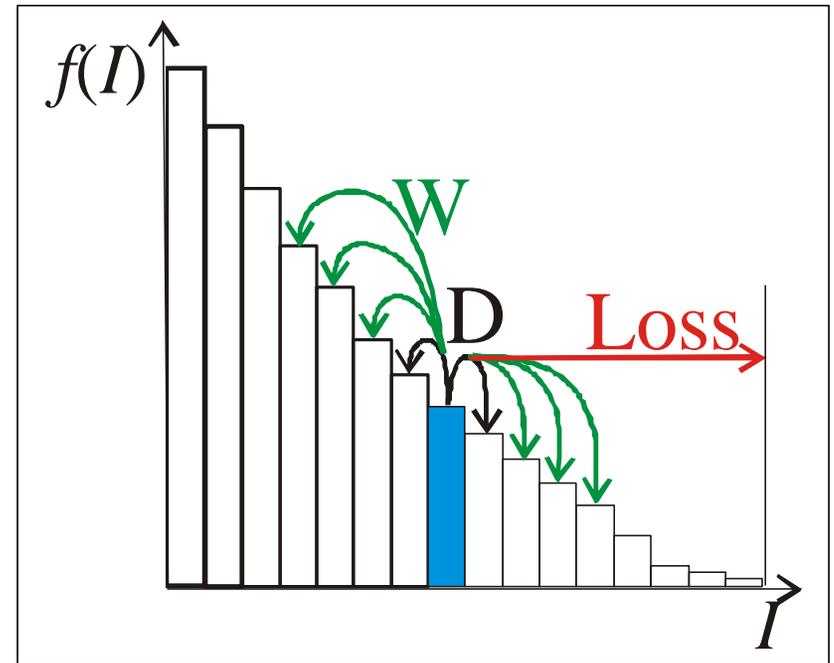
- The kernel is symmetric: $W(I, I') = W(I', I)$,
- The kernel divergence needs to be limited at the minimum action change corresponding to the maximum impact parameter

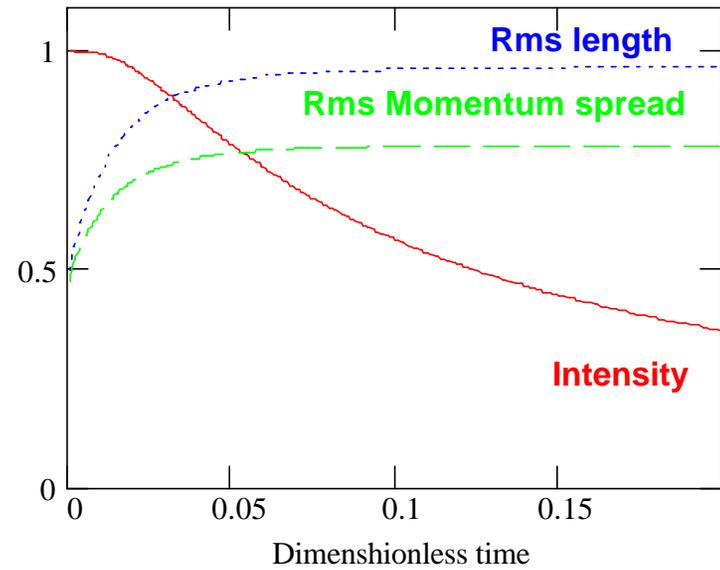
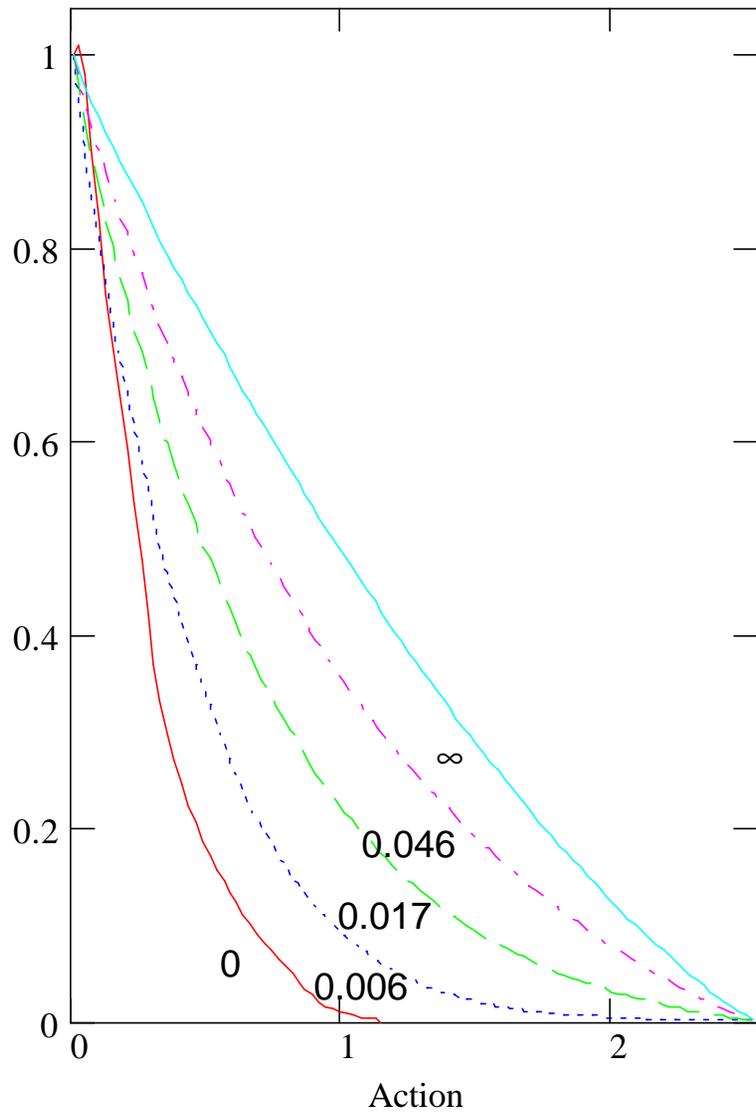
Numerical model

- ◆ Set of bins
 - Transition probabilities
 - Nearby bins – diffusion equation to resolve divergence of $W(I, I')$
 - Far away bins – transition probabilities are described by $W(I, I')$
 - Particle loss outside bucket need to be added
- ◆ In matrix form

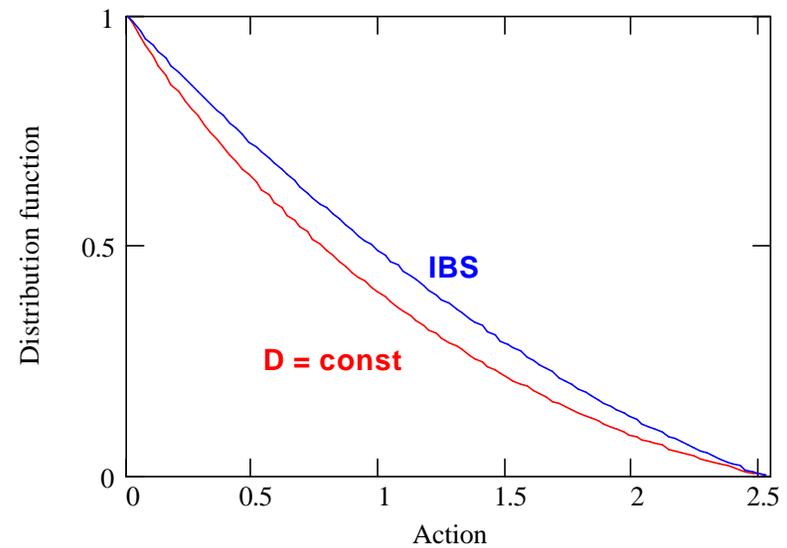
$$\mathbf{f}_{n+1} = \mathbf{f}_n + \mathbf{W}\mathbf{f}_n \Delta t$$

W – is matrix of transition probabilities. It is a symmetric matrix





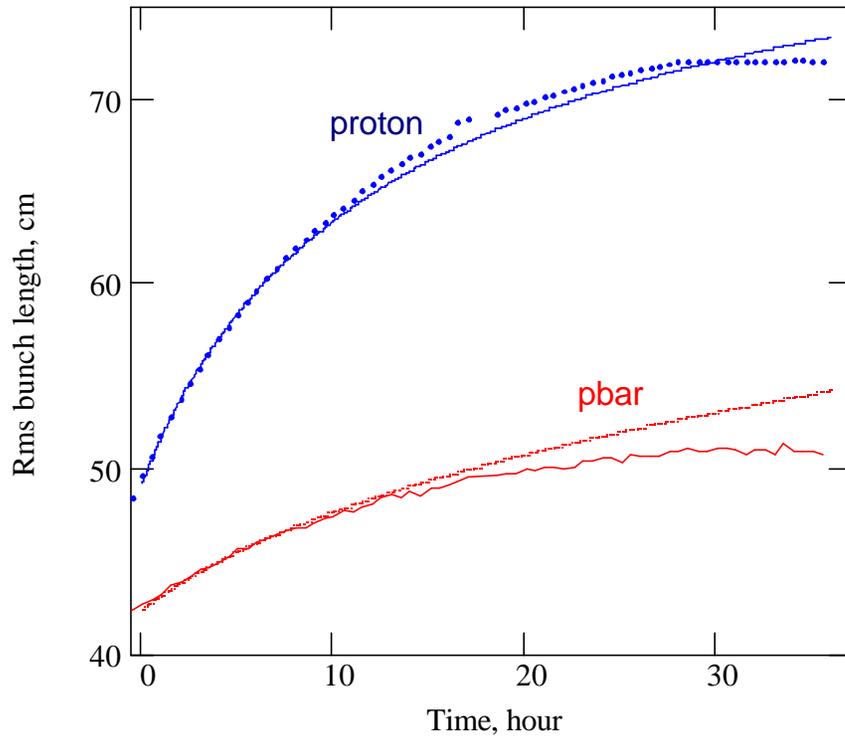
Dependencies: $N(t) / N_0, \sqrt{j^2(t)}, \sqrt{p^2(t)}$



Asymptotic distributions

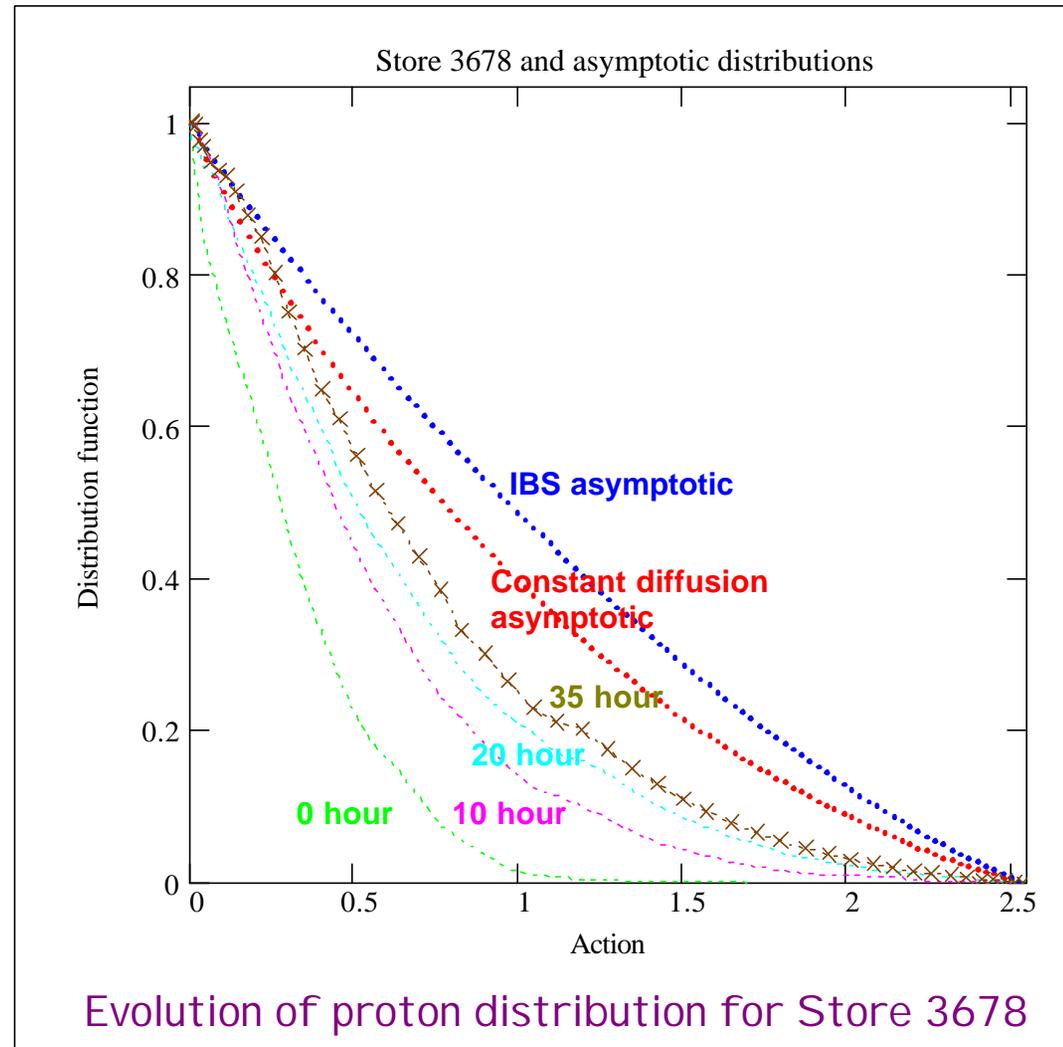
Dependence of longitudinal distribution on time for IBS. Measured initial distribution is used

5. Comparison with experiment

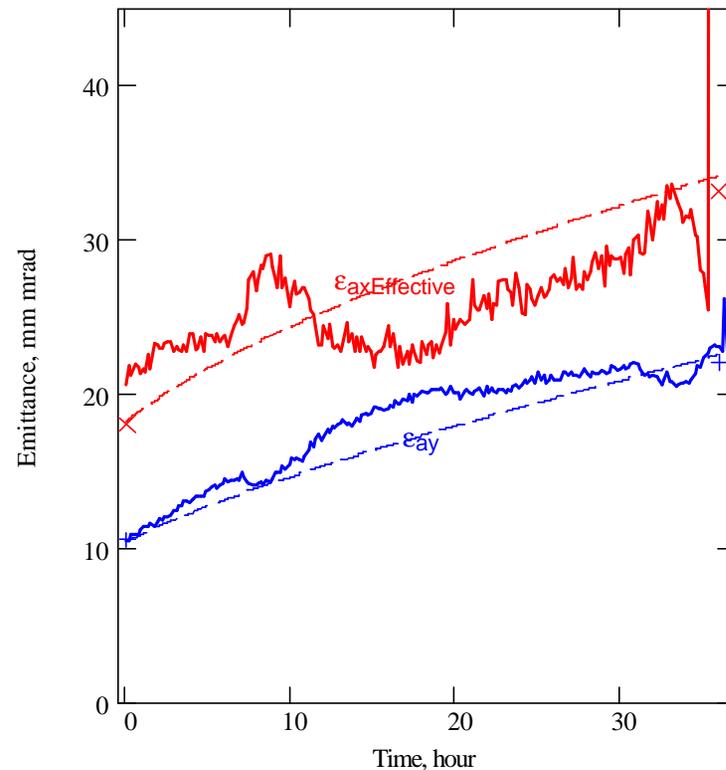
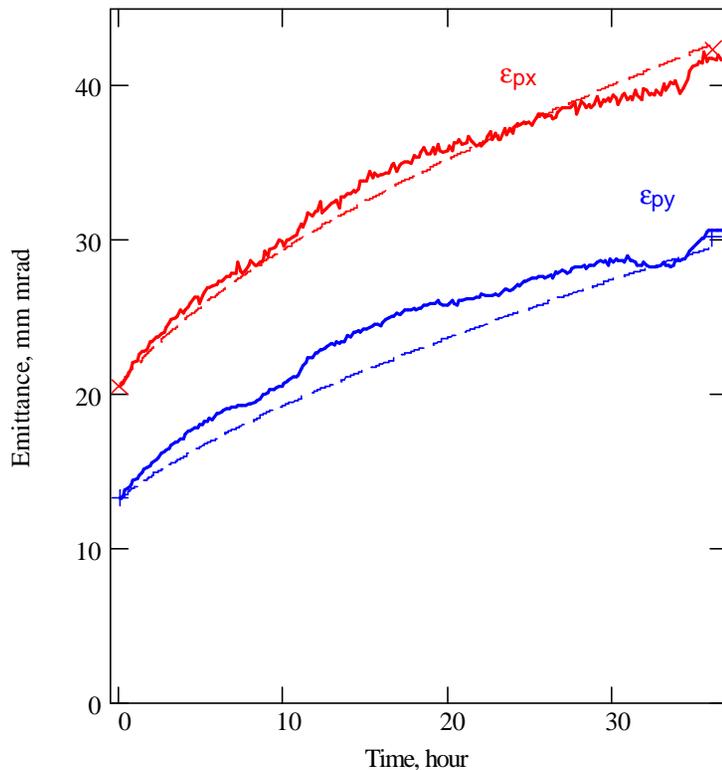


Dependence of computed and measured bunch length on time for Store 3678

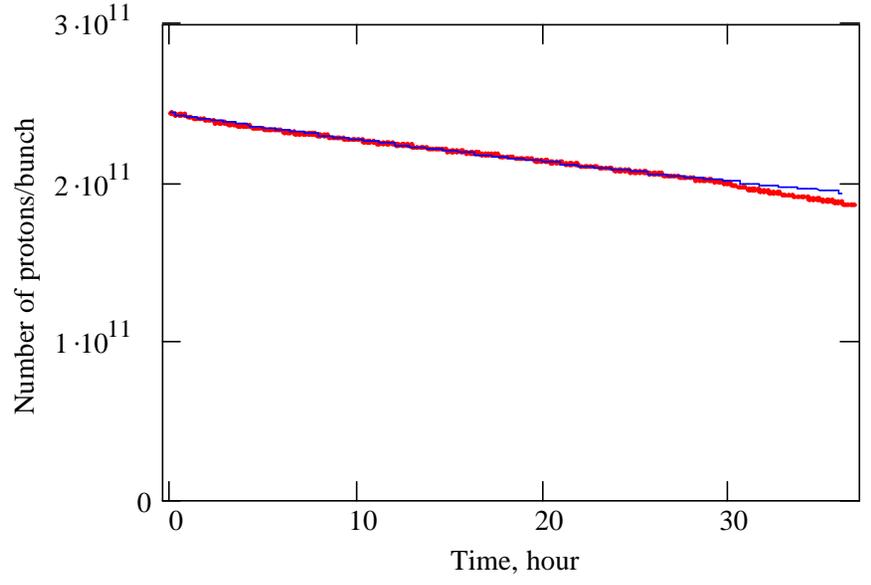
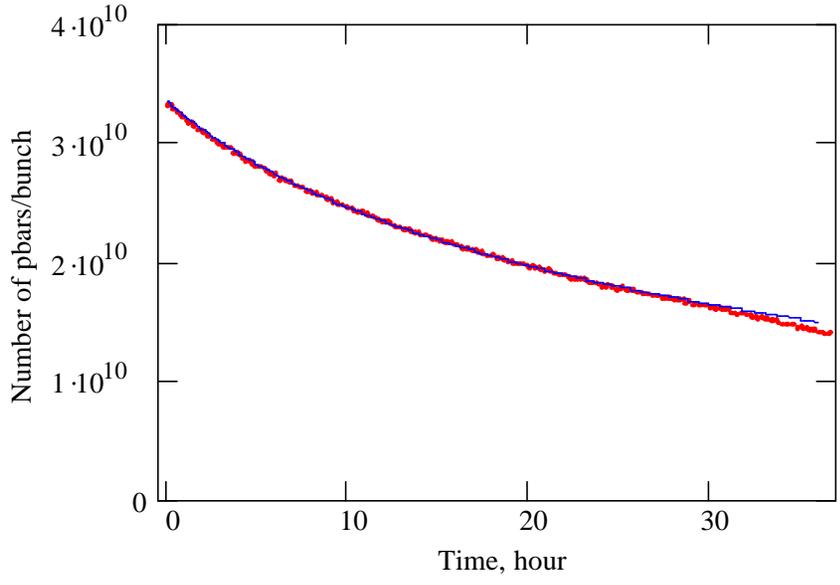
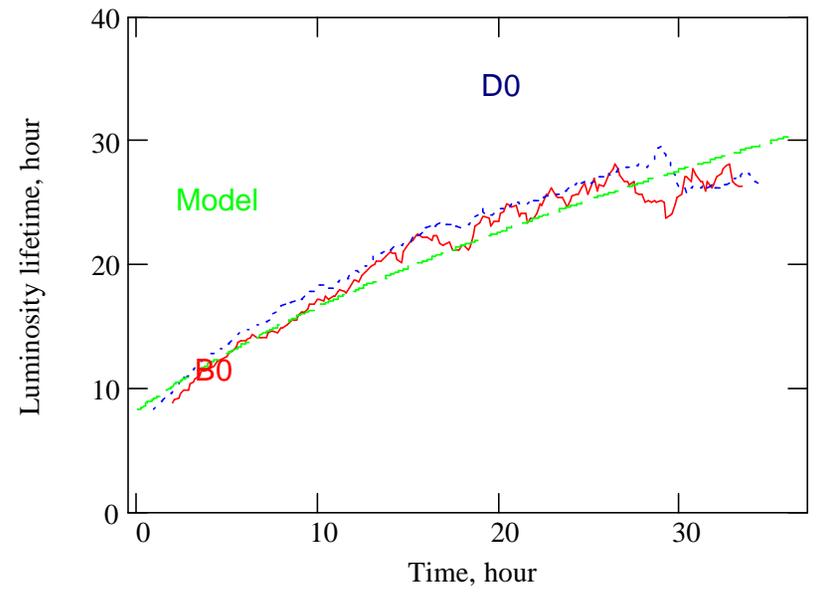
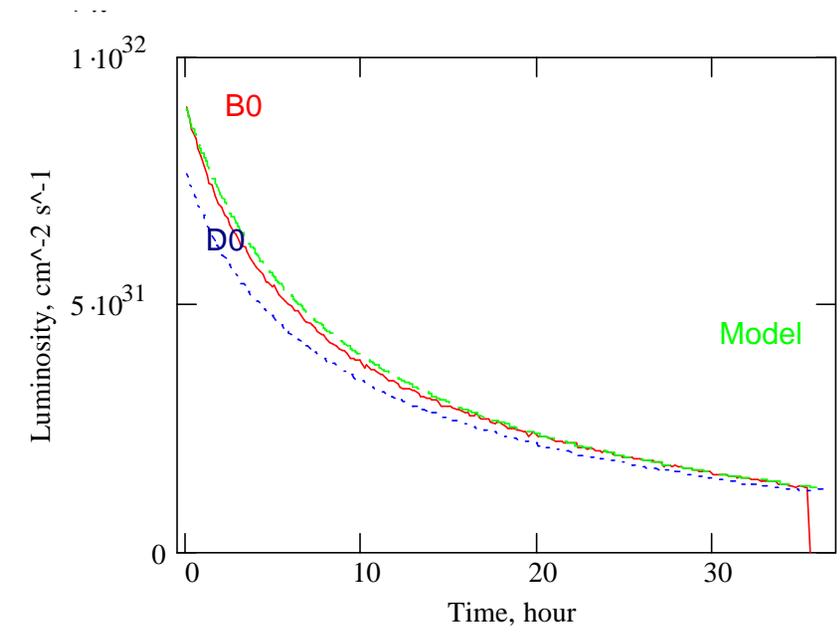
- ◆ Good coincidence for proton bunch lengthening.. It is not affected by choice of free parameters
- ◆ \bar{p} bunch lengthening is affected by RF noise ($\sim 20\%$ of IBS): $S_f = 3 \cdot 10^{11} \text{ rad}^2 \text{ s}$, $d\sigma_s/dt|_{t=0} = 0.12 \text{ cm/hour}$
 - ◆ Sextupole power supply was lost at $t=28$ hour \Rightarrow beam-beam effects



Evolution of proton distribution for Store 3678



- ◆ Both Proton and pbar emittance growths have contributions from scattering at the residual gas. At the store beginning it gives:
 - 2*14% of IBS for protons
 - 2*64% of IBS for pbars
- Gas pressure was set by matching particle loss due to nuclear scattering
- ◆ For pbars there is unaccounted emittance growth (2.4 times of IBS)
 - Beam-beam effects
 - Noise in magnets
 - Preliminary measurements of bunch motion at ω_b yields $\sigma = 0.1 \mu\text{m}$
 - That is consistent with the measurements



Luminosity and beam parameters evolution for Store 3678

Conclusions

- ◆ Theory describes well observed evolution of parameters for proton beam
- ◆ Observed discrepancy for antiprotons is related to other effects which are presently not taken into account
 - Noise in magnets and beam-beam effects are the most probable reasons
- ◆ To get such good agreement the improvements in theory as well as in experiment have been required

Backup transparencies

1. Emittance Growth Rates for X-Y coupled motion in the case of pancake distribution

➤ Growth rates for the momentum spread in the bunched beam

- After averaging over the bunch length and the cross-section

$$\frac{d}{dt}(\mathbf{s}_p^2) \equiv \left\langle \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s = \frac{1}{4\sqrt{2}} \frac{Nr_p^2 c}{\mathbf{g}^3 \mathbf{b}^3} \left\langle \frac{1}{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_s} \frac{\Psi(\mathbf{s}_p / \mathbf{g}, \mathbf{q}_1, \mathbf{q}_2)}{\sqrt{\mathbf{q}_1^2 + \mathbf{q}_2^2 + (\mathbf{s}_p / \mathbf{g})^2}} L_C \right\rangle_s$$

- Here \mathbf{q}_1 and \mathbf{q}_2 are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and \mathbf{s}_1 and \mathbf{s}_2 are ellipse semi-axis in the x-y plane
- This equation is still valid for arbitrary gaussian distribution (pancake distr. is not required)
- For uncoupled beam

$$\mathbf{s}_1 \equiv \mathbf{s}_x = \sqrt{\mathbf{e}_x \mathbf{b}_y + D^2 \mathbf{s}_p^2},$$

$$\mathbf{s}_2 \equiv \mathbf{s}_y = \sqrt{\mathbf{e}_y \mathbf{b}_y},$$

$$\mathbf{q}_1 \equiv \mathbf{q}_x = \sqrt{\frac{\mathbf{e}_x}{\mathbf{b}_x} \left(1 + \frac{(D' \mathbf{b}_x + \mathbf{a}_x D)^2 \mathbf{s}_p^2}{\mathbf{e}_x \mathbf{b}_x + D^2 \mathbf{s}_p^2} \right)},$$

$$\mathbf{q}_2 \equiv \mathbf{q}_y = \sqrt{\mathbf{e}_y / \mathbf{b}_y}$$

➤ Optics is described with Mais-Ripken beta-functions

- For coupled motion the eigen-vectors can be parameterized as

$$\hat{\mathbf{x}}(s) = \text{Re}(\tilde{\mathbf{e}}_1 \mathbf{v}_1(s) e^{-im_1(s)} + \tilde{\mathbf{e}}_2 \mathbf{v}_2(s) e^{-im_2(s)}),$$

$$\mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{\mathbf{b}_{1x}}}{i(1-u) + \mathbf{a}_{1x}} \\ \sqrt{\mathbf{b}_{1x}} \\ \sqrt{\mathbf{b}_{1y}} e^{in_1} \\ -\frac{i u + \mathbf{a}_{1y}}{\sqrt{\mathbf{b}_{1y}}} e^{in_1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{\sqrt{\mathbf{b}_{2x}} e^{in_2}}{-iu + \mathbf{a}_{2x}} \\ \sqrt{\mathbf{b}_{2x}} \\ \sqrt{\mathbf{b}_{2y}} \\ -\frac{i(1-u) + \mathbf{a}_{2y}}{\sqrt{\mathbf{b}_{2y}}} \end{bmatrix}$$

- To find beam sizes and local angular spreads First introduce bilinear form describing the beam ellipse in 4D space

$$\hat{\mathbf{x}}^T \hat{\mathbf{H}} \hat{\mathbf{x}} = 1$$

$$\hat{\mathbf{H}}_{11} = \frac{(1-u)^2 + \mathbf{a}_{1x}^2}{\mathbf{e}_1 \mathbf{b}_{1x}} + \frac{u^2 + \mathbf{a}_{2x}^2}{\mathbf{e}_2 \mathbf{b}_{2x}},$$

$$\hat{\mathbf{H}}_{22} = \frac{\mathbf{b}_{1x}}{\mathbf{e}_1} + \frac{\mathbf{b}_{2x}}{\mathbf{e}_2},$$

$$\hat{\mathbf{H}}_{33} = \frac{u^2 + \mathbf{a}_{1y}^2}{\mathbf{e}_1 \mathbf{b}_{1y}} + \frac{(1-u)^2 + \mathbf{a}_{2y}^2}{\mathbf{e}_2 \mathbf{b}_{2y}},$$

$$\hat{\mathbf{H}}_{44} = \frac{\mathbf{b}_{1y}}{\mathbf{e}_1} + \frac{\mathbf{b}_{2y}}{\mathbf{e}_2},$$

$$\hat{\mathbf{H}}_{12} = \hat{\mathbf{H}}_{21} = \frac{\mathbf{a}_{1x}}{\mathbf{e}_1} + \frac{\mathbf{a}_{2x}}{\mathbf{e}_2},$$

$$\hat{\mathbf{H}}_{34} = \hat{\mathbf{H}}_{43} = \frac{\mathbf{a}_{1y}}{\mathbf{e}_1} + \frac{\mathbf{a}_{2y}}{\mathbf{e}_2},$$

$$\hat{\Gamma}_{13} = \hat{\Gamma}_{31} = \frac{[\mathbf{a}_{1x} \mathbf{a}_{1y} + u(1-u)] \cos \mathbf{n}_1 + [\mathbf{a}_{1y} (1-u) - \mathbf{a}_{1x} u] \sin \mathbf{n}_1}{\mathbf{e}_1 \sqrt{\mathbf{b}_{1x} \mathbf{b}_{1y}}} + \frac{[\mathbf{a}_{2x} \mathbf{a}_{2y} + u(1-u)] \cos \mathbf{n}_2 + [\mathbf{a}_{2x} (1-u) - \mathbf{a}_{2y} u] \sin \mathbf{n}_2}{\mathbf{e}_2 \sqrt{\mathbf{b}_{2x} \mathbf{b}_{2y}}}$$

$$\hat{\Gamma}_{14} = \hat{\Gamma}_{41} = \sqrt{\frac{\mathbf{b}_{1y}}{\mathbf{b}_{1x}}} \frac{\mathbf{a}_{1x} \cos \mathbf{n}_1 + (1-u) \sin \mathbf{n}_1}{\mathbf{e}_1} + \sqrt{\frac{\mathbf{b}_{2y}}{\mathbf{b}_{2x}}} \frac{\mathbf{a}_{2x} \cos \mathbf{n}_2 - u \sin \mathbf{n}_2}{\mathbf{e}_2}$$

$$\hat{\Gamma}_{23} = \hat{\Gamma}_{32} = \sqrt{\frac{\mathbf{b}_{1x}}{\mathbf{b}_{1y}}} \frac{\mathbf{a}_{1y} \cos \mathbf{n}_1 - u \sin \mathbf{n}_1}{\mathbf{e}_1} + \sqrt{\frac{\mathbf{b}_{2x}}{\mathbf{b}_{2y}}} \frac{\mathbf{a}_{2y} \cos \mathbf{n}_2 + (1-u) \sin \mathbf{n}_2}{\mathbf{e}_2}$$

$$\hat{\Gamma}_{24} = \hat{\Gamma}_{42} = \frac{\sqrt{\mathbf{b}_{1x} \mathbf{b}_{1y}} \cos \mathbf{n}_1}{\mathbf{e}_1} + \frac{\sqrt{\mathbf{b}_{2x} \mathbf{b}_{2y}} \cos \mathbf{n}_2}{\mathbf{e}_2}$$

➤ Then we can write the distribution function in the following form

$$f(\hat{\mathbf{x}}, \mathbf{q}_{\parallel}) = C_1 \exp\left(-\frac{1}{2}(\hat{\mathbf{x}} - \mathbf{D}\mathbf{q}_{\parallel})^T \mathbf{?} (\hat{\mathbf{x}} - \mathbf{D}\mathbf{q}_{\parallel})\right) \exp\left(-\frac{\mathbf{q}_{\parallel}^2}{2\mathbf{s}_p^2}\right)$$

where $\mathbf{D} = [D_x \quad D'_x \quad D_y \quad D'_y]^T$

- After integration over momentum spread we obtain

$$f(\hat{\mathbf{x}}) = C_2 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T \mathbf{?}' \hat{\mathbf{x}}\right), \quad \mathbf{?}' = \mathbf{?} - \frac{\mathbf{?} \mathbf{D} \mathbf{D}^T \mathbf{?}}{\mathbf{s}_p^{-2} + \mathbf{D}^T \mathbf{?} \mathbf{D}}$$

➤ Beam sizes

- Size projections

$$\mathbf{s}_x \equiv \sqrt{x^2} = \sqrt{\mathbf{e}_1 \mathbf{b}_{1x} + \mathbf{e}_2 \mathbf{b}_{2x} + D_x^2 \mathbf{s}_p^2}$$

$$\mathbf{s}_y \equiv \sqrt{y^2} = \sqrt{\mathbf{e}_1 \mathbf{b}_{1y} + \mathbf{e}_2 \mathbf{b}_{2y} + D_y^2 \mathbf{s}_p^2}$$

$$\mathbf{a}_{xy} \equiv \frac{\overline{xy}}{\mathbf{s}_x \mathbf{s}_y} = \frac{\mathbf{e}_1 \sqrt{\mathbf{b}_{1x} \mathbf{b}_{1y}} \cos n_1 + \mathbf{e}_2 \sqrt{\mathbf{b}_{2x} \mathbf{b}_{2y}} \cos n_2 + D_x D_y \mathbf{s}_p^2}{\mathbf{s}_x \mathbf{s}_y}$$

- Ellipse semi-axis

$$\mathbf{s}_{1,2} = \sqrt{\frac{2(1 - \mathbf{a}_{xy}^2)}{\mathbf{s}_x^2 + \mathbf{s}_y^2 \pm \sqrt{(\mathbf{s}_x^2 - \mathbf{s}_y^2)^2 + 4\mathbf{a}_{xy}^2 \mathbf{s}_x^2 \mathbf{s}_y^2}}}$$

➤ Local transverse velocity spreads

- Bilinear form for angular spreads

$$[0 \quad \mathbf{q}_x \quad 0 \quad \mathbf{q}_x] \Xi' \begin{bmatrix} 0 \\ \mathbf{q}_x \\ 0 \\ \mathbf{q}_y \end{bmatrix} = \mathbf{q}_x^2 \Xi'_{22} + 2\mathbf{q}_x \mathbf{q}_y \Xi'_{24} + \mathbf{q}_y^2 \Xi'_{44} = 1$$

- Ellipse semi-axis in the plane of local angular spreads (x'-y' plane)

$$\mathbf{q}_{1,2} = \sqrt{\frac{2}{\Xi'_{22} + \Xi'_{44} \pm \sqrt{(\Xi'_{22} - \Xi'_{44})^2 + 4\Xi'_{24}{}^2}}}$$

➤ Additional transverse emittance growth due to finite dispersion

- For uncoupled motion

$$\frac{d\mathbf{e}_x}{dt} = \left\langle A_x \frac{d}{dt} \left(\frac{\overline{\Delta p_{\parallel}^2}}{p} \right) \right\rangle_s, \quad A_x = \frac{D^2 + (D'\mathbf{b}_x + \mathbf{a}_x D)^2}{\mathbf{b}_x}$$

- Coupled motion: momentum change excites both hor. and vert. motions

$$\begin{bmatrix} D_x \\ D'_x \\ D_y \\ D'_y \end{bmatrix} \frac{\Delta p}{p} \equiv \mathbf{D} \frac{\Delta p}{p} = \text{Re}(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2) = \mathbf{V} \mathbf{a}, \quad V = [\mathbf{v}'_1, -\mathbf{v}''_1, \mathbf{v}'_2, -\mathbf{v}''_2], \quad \mathbf{a} = \begin{bmatrix} a'_1 \\ a''_1 \\ a'_2 \\ a''_2 \end{bmatrix}$$

$$\mathbf{a} = \frac{\Delta p}{p} \mathbf{V}^{-1} \mathbf{D}$$

- Then the emittance growth is

$$\frac{d\mathbf{e}_1}{dt} = \left\langle A_1 \frac{d}{dt} \left(\frac{\overline{\Delta p}}{p} \right)^2 \right\rangle_s, \quad \frac{d\mathbf{e}_2}{dt} = \left\langle A_2 \frac{d}{dt} \left(\frac{\overline{\Delta p}}{p} \right)^2 \right\rangle_s$$

$$\text{Here: } A_{1,2} = \mathbf{D}^T \mathbf{B}_{1,2} \mathbf{D}, \quad \mathbf{B}_1 = (\mathbf{V}^{-1})^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{V}^{-1}, \quad \mathbf{B}_2 = (\mathbf{V}^{-1})^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{V}^{-1}$$

- Expressing matrix V through beta-functions we finally obtain

$$\mathbf{B}_1 = \begin{bmatrix} \frac{(1-u)^2 + \mathbf{a}_{1x}^2}{\mathbf{b}_{1x}} & \mathbf{a}_{1x} & B_{113} & B_{114} \\ \mathbf{a}_{1x} & \mathbf{b}_{1x} & B_{123} & \sqrt{\mathbf{b}_{1x}\mathbf{b}_{1y}} \cos n_1 \\ B_{113} & B_{123} & \frac{u^2 + \mathbf{a}_{1y}^2}{\mathbf{b}_{1y}} & \mathbf{a}_{1y} \\ B_{114} & \sqrt{\mathbf{b}_{1x}\mathbf{b}_{1y}} \cos n_1 & \mathbf{a}_{1y} & \mathbf{b}_{1y} \end{bmatrix}$$

$$B_{113} = \frac{(u(1-u) + \mathbf{a}_{1x}\mathbf{a}_{1y})\cos n_1 + (\mathbf{a}_{1y}(1-u) - \mathbf{a}_{1x}u)\sin n_1}{\sqrt{\mathbf{b}_{1x}\mathbf{b}_{1y}}}$$

$$B_{114} = \sqrt{\frac{\mathbf{b}_{1y}}{\mathbf{b}_{1x}}} (\mathbf{a}_{1x} \cos n_1 + (1-u) \sin n_1)$$

$$B_{123} = \sqrt{\frac{\mathbf{b}_{1x}}{\mathbf{b}_{1y}}} (\mathbf{a}_{1y} \cos n_1 - u \sin n_1)$$

$$\mathbf{B}_2 = \begin{bmatrix} \frac{u^2 + \mathbf{a}_{2x}^2}{\mathbf{b}_{2x}} & \mathbf{a}_{2x} & B_{213} & B_{214} \\ \mathbf{a}_{2x} & \mathbf{b}_{2x} & B_{223} & \sqrt{\mathbf{b}_{2x}\mathbf{b}_{2y}} \cos n_2 \\ B_{213} & B_{223} & \frac{(1-u)^2 + \mathbf{a}_{2y}^2}{\mathbf{b}_{2y}} & \mathbf{a}_{2y} \\ B_{214} & \sqrt{\mathbf{b}_{2x}\mathbf{b}_{2y}} \cos n_2 & \mathbf{a}_{2y} & \mathbf{b}_{2y} \end{bmatrix}$$

$$B_{213} = \frac{(u(1-u) + \mathbf{a}_{2x}\mathbf{a}_{2y}) \cos n_2 + (\mathbf{a}_{2x}(1-u) - \mathbf{a}_{2y}u) \sin n_2}{\sqrt{\mathbf{b}_{2x}\mathbf{b}_{2y}}}$$

$$B_{214} = \sqrt{\frac{\mathbf{b}_{2y}}{\mathbf{b}_{2x}}} (\mathbf{a}_{2x} \cos n_2 - u \sin n_2)$$

$$B_{223} = \sqrt{\frac{\mathbf{b}_{2x}}{\mathbf{b}_{2y}}} (\mathbf{a}_{2y} \cos n_2 + (1-u) \sin n_2)$$

- Finally, for ultra-relativistic beam ($\mathbf{g} \gg Q_x, Q_y$), we obtain

$$\frac{d}{dt} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}_s = \frac{1}{4\sqrt{2}} \frac{Nr_p^2 c}{\mathbf{g}^3 \mathbf{b}^3} \left\langle \frac{1}{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_s} \frac{L_C}{\sqrt{\mathbf{q}_1^2 + \mathbf{q}_2^2 + (\mathbf{s}_p / \mathbf{g})^2}} \begin{bmatrix} \hat{A}_1 \Psi(\mathbf{s}_p / \mathbf{g}, \mathbf{q}_1, \mathbf{q}_2) \\ \hat{A}_2 \Psi(\mathbf{s}_p / \mathbf{g}, \mathbf{q}_1, \mathbf{q}_2) \end{bmatrix} \right\rangle_s$$