Stochastic Cooling with Schottky Band Overlap[1]

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Abstract

Abstract. Optimal use of stochastic cooling is essential to maximize the antiproton stacking rate for Tevatron Run II. Good understanding and characterization of the cooling is important for the optimization. The paper is devoted to derivation of the Fokker-Planck equations justified in the case of near or full Schottky base overlap for both longitudinal and transverse coolings.

Introduction

The Schottky band overlap compromises the performance of stochastic cooling. Therefore all practical cooling systems are designed and built to avoid the band overlap. Nevertheless, operating cooling systems are frequently used in a regime when bands are close to overlap or slightly overlapped. In this case the band overlap needs to be taken into account if detailed description of the cooling is required. The stochastic cooling theory with no band overlap is well developed [see Ref. 1 and 2 and included bibliography]. In this paper we extend this theory to the case of arbitrary band overlap. First, we derive expressions for the beam permeabilities of the longitudinal and transverse coolings and, then, proceed to derivation of the Fokker-Planck equations describing transverse and longitudinal coolings.

1. Beam Permeability for Longitudinal Cooling

Usually, a calculation of the beam permeability is based on azimuthal harmonics. It does not work well if bands are close being overlapped because the amplitudes of the harmonics are changed within one revolution. In this paper we limit ourselves to the case of the beam with sufficiently small intensity so that the beam interaction with vacuum chamber could be neglected. That allows us to reduce the problem from one of finding the entire ring distribution function to one of finding the local distribution functions in the pickup and kicker. Figure 1 depicts a layout of the cooling system. Let $f_1(x,t)$ be the distribution function immediately after the kicker, $f_2(x,t)$ be the distribution function in the pickup, and $f_3(x,t)$ be the distribution function just before the kicker. Taking into account that the particle momentum is changed only in the kicker one can write the equations binding up these functions:

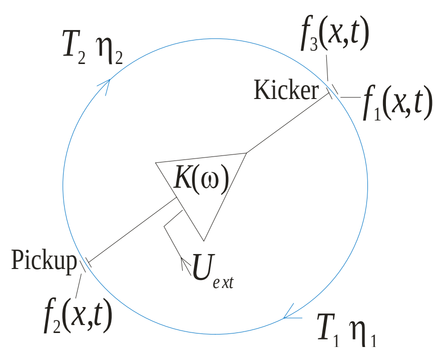

Figure 1. Schematic of longitudinal cooling system.

[1] This paper represents an extended version of the paper presented for COOL-2005 conference, Galena, IL, USA, Sep. 2005.

$$
\begin{aligned}
f_2(x,t) &= f_1\left(x, t - T_1(1+\eta_1 x_0)\right) \quad , \\
f_3(x,t) &= f_2\left(x, t - T_2(1+\eta_2 x)\right) \quad , \\
f_1(x,t) &= f_3\left(x - \delta p(t)/p_0, t)\right) \quad .
\end{aligned}
\tag{1}
$$

Here $x = (p - p_0)/p_0$ is the relative momentum deviation, T_1, T_2 and $T_0 = T_1 + T_2$ are the kicker-to-pickup, pickup-to-kicker and revolution times for the reference particle, $\eta = \alpha - 1/\gamma^2$ is the slip factor, η_1 and η_2 are the partial kicker-to-pickup and pickup-to-kicker slip factors so that $\eta T_0 = \eta_1 T_1 + \eta_2 T_2$, and $\delta p(t)$ is the particle momentum change by the kicker. Expressing the distribution function through its equilibrium value and the perturbation, $f_k(x,t) = f_0(x) + \tilde{f}_k(x,t),\ k = 1,...3$, and leaving only the first order addend in the Taylor expansion of the third equation in Eq. (1) one obtains:

$$
\begin{aligned}
\tilde{f}_2(x,t) &= \tilde{f}_1\left(x, t - T_1(1+\eta_1 x_0)\right) \quad , \\
\tilde{f}_3(x,t) &= \tilde{f}_2\left(x, t - T_2(1+\eta_2 x)\right) \quad , \\
\tilde{f}_1(x,t) &= \tilde{f}_3(x,t) - \frac{\delta p(t)}{p_0}\frac{df_0(x)}{dx} \quad .
\end{aligned}
\tag{2}
$$

We will look for a solution in the form $\tilde{f}_k(x,t) = \tilde{f}_{k\omega}(x)e^{i\omega t}$ and $\delta p(t) = \delta p_\omega e^{i\omega t}$. Substituting these equations in Eq. (2) we obtain:

$$
\begin{aligned}
\tilde{f}_{2\omega}(x) &= \tilde{f}_{1\omega}(x)\exp\left(-i\omega T_1(1+\eta_1 x)\right) \quad , \\
\tilde{f}_{3\omega}(x) &= \tilde{f}_{2\omega}(x)\exp\left(-i\omega T_2(1+\eta_2 x)\right) \quad , \\
\tilde{f}_{1\omega}(x) &= \tilde{f}_{3\omega}(x) - \frac{df_0(x)}{dx}\frac{\delta p_\omega}{p_0} \quad .
\end{aligned}
\tag{3}
$$

Excluding $\tilde{f}_{1\omega}(x)$ and $\tilde{f}_{3\omega}(x)$ from the above equations we obtain:

$$
\tilde{f}_{2\omega}(x)\exp\left(i\omega T_1(1+\eta_1 x)\right) = \tilde{f}_{2\omega}(x)\exp\left(-i\omega T_2(1+\eta_2 x)\right) - \frac{df_0(x)}{dx}\frac{\delta p_\omega}{p_0} \quad . \tag{4}
$$

Let the momentum kick be determined by the sum of amplified pickup signal and an external harmonic perturbation so that:

$$
\delta p_\omega / p_0 = \int dx\, \tilde{f}_{2\omega}(x) G(x,\omega) e^{-i\omega T_2}\left[1 - A(\omega)e^{-i\omega T_0}\right] + \Delta p_{ext\omega}/p_0 \quad . \tag{5}
$$

Here the term $e^{-i\omega T_2}$ takes into account the delay in signal propagation from the pickup to the kicker, $\tilde{f}_2(p, t - T_2) \to \tilde{f}_{2\omega}(p)e^{-i\omega T_2}$. The total system gain, $G(x,\omega)\left[1 - A(\omega)e^{-i\omega T_0}\right]$, is chosen so that it would describe both Palmer and momentum cooling. For Palmer cooling $A(\omega)$ = 0 and the pickup signal depends on the particle momentum due to non-zero dispersion in the pickup. For filter cooling the pickup signal does not depend on particle momentum, $G(x,\omega) \to G(\omega)$, and the cooling signal is formed by the notch filter, $A(\omega) \approx 1$. Its delay is equal to the revolution time for the reference particle, T_0. Taking into account the distribution function normalization, $\int f_0(x)dx = 1$, and introducing the impedances of pickup, Z_p, and kicker, Z_k, so that the pickup voltage is

$$
U_{pickup\,\omega} = I_0 \int Z_p(x,\omega) f_{2\omega}(x,\omega) dx \quad , \tag{6}
$$

and the energy gain in the kicker is

$$\delta E_{kic\,\mathrm{ker}_\omega} = e\frac{Z_k(\omega)}{Z_{ampl}}U_{kic\,\mathrm{ker}_\omega} \quad , \tag{7}$$

we obtain that the system gain is:

$$G(x,\omega) = \frac{eI_0 Z_p(x,\omega)Z_k(\omega)}{\gamma\beta^2 mc^2 Z_{ampl}}K(\omega) \quad . \tag{8}$$

Here I_0 is the beam current, Z_{ampl} = 50 Ω is the impedance of power amplifier, $K(\omega)$ is the total electronic amplification of the cooling system, c is the speed of the light, e and m are the particle charge and mass, and β and γ are the relativistic factors.

Substitution Eq. (5) into Eq. (4) yields:

$$\begin{aligned}&\tilde{f}_{2\omega}(x)\left[e^{i\omega T_1(1+\eta_1 x)} - e^{-i\omega T_2(1+\eta_2 x)}\right] + \\ &\frac{df_0(x)}{dx}\left[\frac{\Delta p_{ext\omega}}{p_0} + e^{-i\omega T_2}\left[1 - A(\omega)e^{-i\omega T_0}\right]\int dx' \tilde{f}_{2\omega}(x')G(x',\omega)\right] = 0 \quad .\end{aligned} \tag{9}$$

Dividing both addends by $e^{i\omega T_1(1+\eta_1 x)} - e^{-i\omega T_2(1+\eta_2 x)}$, multiplying them by $G(x,\omega)$ and integrating we obtain:

$$\begin{aligned}&S_\omega + \frac{\Delta p_{ext\omega}}{p_0}\int\frac{df_0(x)}{dx}\frac{G(x,\omega)dx}{e^{i\omega T_1(1+\eta_1 x)} - e^{-i\omega T_2(1+\eta_2 x)}} + \\ &e^{-i\omega T_2}\left[1 - A(\omega)e^{-i\omega T_0}\right]S_\omega\int\frac{df_0(x)}{dx}\frac{G(x,\omega)dx}{e^{i\omega T_1(1+\eta_1 x)} - e^{-i\omega T_2(1+\eta_2 x)}} = 0 \quad ,\end{aligned} \tag{10}$$

where

$$S_\omega = \int dx' \tilde{f}_{2\omega}(x')G(x',\omega). \tag{11}$$

Solving Eq. (10) relative to S_ω we finally obtain the system response at the pickup location due to the external harmonic perturbation:

$$S_\omega = -\frac{1}{\varepsilon(\omega)}\frac{\Delta p_{ext\omega}}{p_0}\int_{\delta\to 0_+}\frac{df_0(x)}{dx}\frac{G(x',\omega)e^{i\omega T_2(1+\eta_2 x)}}{e^{i\omega T_0(1+\eta x)} - (1-\delta)}dx \quad , \tag{12}$$

where $\varepsilon(\omega)$ is the beam permeability

$$\varepsilon(\omega) = 1 + \left(1 - A(\omega)e^{-i\omega T_0}\right)\int_{\delta\to 0_+}\frac{df_0(x)}{dx}\frac{G(x,\omega)e^{i\omega T_2\eta_2 x}}{e^{i\omega T_0(1+\eta x)} - (1-\delta)}dx \quad . \tag{13}$$

In the above equations the rule to traverse the poles, $\delta \to 0_+$, follows from the fact that for the complex Laplace transform ω is shifted to the lower complex plane.

Far away from Schottky band overlap the exponent in the denominator of Eq. (13) can be expanded near revolution harmonic, $\omega = n\omega_0 + \delta\omega$, $\omega_0 = 2\pi / T_0$ and we arrive to the standard formula for the permeability[1]:

$$\varepsilon_n(\delta\omega) = 1 + \left(1 - A(\omega)e^{-i\omega T_0}\right)\int_{\delta\to 0_+}\frac{df_0(x)}{dx}\frac{G(x,\omega)}{2\pi i\left(\eta n x + \delta\omega / \omega_0 - i\delta\right)}dx \quad . \tag{14}$$

To find the closed system response we need to restore in Eq. (12) two missed factors. They describe the pickup-to-kicker delay, $e^{-i\omega T_2}$, and the notch filter. That results in:

$$S_{\omega_{closed}} = -\frac{1}{\varepsilon(\omega)}\frac{\Delta p_{ext\omega}}{p_0}\left(1 - A(\omega)e^{-i\omega T_0}\right)\int_{\delta\to 0_+}\frac{df_0(x)}{dx}\frac{G(x,\omega)e^{i\omega T_2\eta_2 x}}{e^{i\omega T_0(1+\eta x)} - (1-\delta)}dx \quad , \tag{14a}$$

The response of the open system can be obtained from Eq. (14a) by setting $\varepsilon(\omega)$=1.

2. Beam Permeability for Transverse Cooling

Similar to the method used above for the longitudinal cooling the beam evolution is considered at three points: (1) after kicker, (2) in the pickup, and (3) before the kicker. The layout of the system is presented in Figure 2. The beam dipole moment at each point is

$$d_k(t) = \frac{I_0}{c\beta}\int y_k(x,t) f_0(x) dx \quad , \quad k = 1,2,3. \tag{15}$$

Here $f_0(x)$ is the distribution function over momentum, and $y_k(x)$ is the average transverse beam displacement for particles with relative momentum deviations equal to x. Normalizing the beam displacements, $y_k(x)$, and angles, $\theta_k(x)$, by the beta-functions so that $\tilde{y}_k = y_k / \sqrt{\beta_k}$ and $\tilde{\theta}_k = \theta_k\sqrt{\beta_k} + \alpha_k x_k / \sqrt{\beta_k}$ one can write the system of equations binding up the beam displacements after and before the kicker:

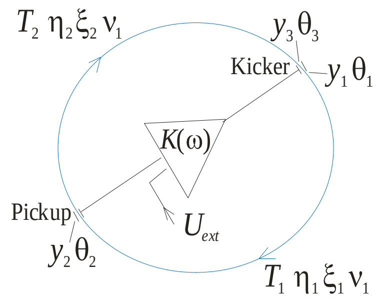

Figure 2. Schematic of transverse cooling system.

$$\begin{aligned} \tilde{y}_3(x,t) &= c(x)\tilde{y}_1(x, t - T_0(1+\eta x)) + s(x)\tilde{\theta}_1(x, t - T_0(1+\eta x)) \quad , \\ \tilde{\theta}_3(x,t) &= -s(x)\tilde{y}_1(x, t - T_0(1+\eta x)) + c(x)\tilde{\theta}_1(x, t - T_0(1+\eta x)) \quad . \end{aligned} \tag{16}$$

Here $c(x) = \cos(2\pi(\nu + \xi x))$, $s(x) = \sin(2\pi(\nu + \xi x))$, ν is the betatron tune, and ξ is the tune chromaticity. Passing the kicker changes the beam angle but does not change beam coordinate so that

$$\begin{aligned} \tilde{y}_1(x,t) &= \tilde{y}_3(x,t) \quad , \\ \tilde{\theta}_1(x,t) &= \tilde{\theta}_3(x,t) + \delta\tilde{\theta}(t) \quad . \end{aligned} \tag{17}$$

We look for a solution in the form $\tilde{y}_k(x,t) = \tilde{y}_{k\omega}(x)e^{i\omega t}$ and $\delta\theta(t) = \delta\theta_\omega e^{i\omega t}$. Substituting it into Eqs. (16) and (17) we obtain:

$$\begin{aligned} \tilde{y}_{3\omega}(x) &= \left(c(x)\tilde{y}_{1\omega}(x) + s(x)\tilde{\theta}_{1\omega}(x)\right)\exp(-i\omega T_0(1+\eta x)) \quad , \\ \tilde{\theta}_{3\omega}(x) &= \left(-s(x)\tilde{y}_{1\omega}(x) + c(x)\tilde{\theta}_{1\omega}(x)\right)\exp(-i\omega T_0(1+\eta x) \quad , \\ \tilde{y}_{1\omega}(x) &= \tilde{y}_{3\omega}(x) \quad , \\ \tilde{\theta}_{1\omega}(x) &= \tilde{\theta}_{3\omega}(x) + \delta\tilde{\theta}_\omega \quad . \end{aligned} \tag{18}$$

Solution these equations relative to $\tilde{y}_{1\omega}(x)$ and $\tilde{\theta}_{1\omega}(x)$ yields:

$$\begin{aligned} \tilde{\theta}_{1\omega}(x) &= -\frac{\left(c(x) - \exp(i\omega T_0(1+\eta x))\right)\exp(i\omega T_0(1+\eta x))}{\exp(2i\omega T_0(1+\eta x)) - 2c(x)\exp(i\omega T_0(1+\eta x)) + 1}\delta\tilde{\theta}_\omega \quad , \\ \tilde{y}_{1\omega}(x) &= \frac{s(x)\exp(i\omega T_0(1+\eta x))}{\exp(2i\omega T_0(1+\eta x)) - 2c(x)\exp(i\omega T_0(1+\eta x)) + 1}\delta\tilde{\theta}_\omega \quad . \end{aligned} \tag{19}$$

Taking into account the relationship between coordinates and angles of points 1 and 2,

$$\tilde{y}_2(x,t) = c_1(x)\tilde{y}_1(x,t - T_1(1+\eta_1 x)) + s_1(x)\tilde{\theta}_1(x,t - T_1(1+\eta_1 x)) \quad ,$$
$$\tilde{\theta}_2(x,t) = -s_1(x)\tilde{y}_1(x,t - T_1(1+\eta_1 x)) + c_1(x)\tilde{\theta}_1(x,t - T_1(1+\eta_1 x)) \quad , \tag{20}$$

and transforming the time dependent values in Eq. (16) to their Fourier harmonics we obtain for the beam displacement in the pickup

$$\tilde{y}_{2\omega}(x) = \frac{\left(s_2(x) + s_1(x)e^{i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1}\delta\tilde{\theta}_\omega \quad . \tag{21}$$

Here $c_{1,2}(x) = \cos\left(2\pi(\nu_{1,2} + \xi_{1,2}x)\right)$, $s_{1,2}(x) = \sin\left(2\pi(\nu_{1,2} + \xi_{1,2}x)\right)$, $2\pi\nu_1$ and $2\pi\nu_2$ are the betatron phase advances between pickup and kicker so that $\nu_1 + \nu_2 = \nu$, and ξ_1 and ξ_2 are the partial tune chromaticities so that $\xi_1 + \xi_2 = \xi$.

Similar to Eq. (5) the beam kick is determined by the sum of amplified pickup signal and an external harmonic perturbation so that:

$$\delta\tilde{\theta}_\omega = \int dx\, f_0(x)\tilde{y}_{2\omega}(x)G_\perp(\omega)e^{-i\omega T_2} + \Delta\tilde{\theta}_{ext\omega} \quad . \tag{22}$$

We introduce the impedances of pickup, $Z_{p\perp}$, and kicker, $Z_{k\perp}$, so that the pickup voltage is

$$U_{pickup\,\omega} = I_0 Z_{p\perp}(\omega)\overline{y_\omega} = I_0 Z_{p\perp}(\omega)\int y_\omega(x) f_0(x)dx \quad , \tag{23}$$

and the transverse angle obtained by a particle in the kicker is

$$\delta\theta_{kicker_\omega} = \frac{e}{mc^2\gamma\beta^2}\frac{Z_{k\perp}(\omega)}{Z_{ampl}}U_{kicker_\omega} \quad . \tag{24}$$

That yields that the system gain is:

$$G_\perp(\omega) = \frac{eI_0 Z_{p\perp}(\omega)Z_{k\perp}(\omega)}{\gamma\beta^2 mc^2 Z_{ampl}}\sqrt{\beta_p\beta_k}K(\omega) \quad , \tag{25}$$

where β_p and β_k are the beta-functions in the pickup and kicker.

Substituting Eq. (22) into Eq. (21) we obtain:

$$\tilde{y}_{2\omega}(x) = \frac{\left(s_2(x) + s_1(x)e^{2i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1}\left(G_\perp(\omega)\,e^{-i\omega T_2}\int dx\, f_0(x)\tilde{y}_{2\omega}(x) + \Delta\tilde{\theta}_{ext\omega}\right). \tag{26}$$

The solution is similar to the solution carried out in the previous section. The result is:

$$\overline{\tilde{y}_{2\omega}} \equiv \int dx\, f_0(x)\tilde{y}_{2\omega}(x) = \frac{\Delta\tilde{\theta}_{ext\omega}}{\varepsilon_\perp(\omega)}\int dx\, f_0(x)\frac{\left(s_2(x) + s_1(x)e^{i\omega T_0(1+\eta x)}\right)e^{i\omega T_2(1+\eta_2 x)}}{e^{2i\omega T_0(1+\eta x)} - 2c(x)e^{i\omega T_0(1+\eta x)} + 1} \quad . \tag{27}$$

where the beam permeability is:

$$\varepsilon_\perp(\omega) = 1 - \frac{G_\perp(\omega)}{2}\int\limits_{\delta\to 0_+}\frac{\left[e^{-i\omega T_0(1+\eta x)}\sin\left(2\pi(\nu_2 + \xi_2 x)\right) + \sin\left(2\pi(\nu_1 + \xi_1 x)\right)\right]e^{i\omega T_2\eta_2 x}}{\cos\left(\omega T_0(1+\eta x)\right) - \cos\left(2\pi(\nu + \xi x)\right) + i\delta\sin\left(\omega T_0(1+\eta x)\right)}f_0(x)dx \,. \tag{28}$$

and the rule to traverse the poles is determined similar to the Eqs. (12) and (13).

Far away from Schottky band overlap the cosines in the denominator of Eq. (28) can be expended near betatron sideband, $\omega = (n+\nu)\omega_0 + \delta\omega$, $\omega_0 = 2\pi/T_0$ and we arrive to the standard formula for the transverse permeability[1]:

Far away from Schottky band overlap the cosines in the denominator of Eq. (28) can be expended near betatron sidebands, $\omega_{n\pm} = (n \pm \nu)\omega_0$, $\omega_0 = 2\pi/T_0$ and we arrive to the standard formula for the transverse permeability:

$$\varepsilon_{\perp n\pm}(\delta\omega_{n\pm}) = 1 + \frac{G_\perp(\omega_n)}{4\pi} \int_{\delta\to 0_+} \frac{\sin(2\pi\nu_2)\exp(\mp 2\pi i\nu) + \sin(2\pi\nu_1)}{\left((\eta n \mp \xi)x + \delta\omega_{n\pm}/\omega_0 - i\delta\right)\sin(2\pi\nu)} f_0(x)dx$$
$$\xrightarrow{2\pi\nu_2=\pi/2} 1 + \frac{G_\perp(\omega_n)}{4\pi i} \int_{\delta\to 0_+} \frac{f_0(x)dx}{(\eta n \mp \xi)x + \delta\omega_{n\pm}/\omega_0 - i\delta} \quad , \tag{29}$$

where $\delta\omega_{n\pm} = \omega - \omega_{n\pm}$.

To find the closed system response we need to restore in Eq. (27) the missed factor describing the pickup-to-kicker delay, $e^{-i\omega T_2}$. That results in:

$$\overline{\tilde{y}_{2\omega_{closed}}} = \frac{\Delta\tilde{\theta}_{ext\omega}}{2\varepsilon_\perp(\omega)} \int_{\delta\to 0_+} \frac{\left[e^{-i\omega T_0(1+\eta x)}\sin\left(2\pi(\nu_2+\xi_2 x)\right) + \sin\left(2\pi(\nu_1+\xi_1 x)\right)\right] e^{i\omega T_2\eta_2 x}}{\cos\left(\omega T_0(1+\eta x)\right) - \cos\left(2\pi(\nu+\xi x)\right) + i\delta\sin\left(\omega T_0(1+\eta x)\right)} f_0(x)dx \ . \tag{29a}$$

The response of the open system can be obtained from Eq. (29a) by setting $\varepsilon_\perp(\omega)=1$.

3. Fokker-Planck Equation for Longitudinal Cooling

Evolution of the beam longitudinal distribution function is described by the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(F(x)f\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right) \quad . \tag{30}$$

The drag force is created by the particle self-interaction and therefore is not directly affected by the band overlap but it is suppressed by screening of the particle signal. The result is well-known[1]:

$$F(x) \equiv \frac{dx}{dt} = \frac{1}{T}\sum_{n=-\infty}^{\infty} \frac{G_1(x,\omega_n)}{\varepsilon(\omega_n)}\left(1 - A(\omega_n)e^{-i\omega_n T_0}\right)e^{i\omega_n T_2\eta_2 x}, \quad \omega_n = n\omega_0(1-\eta x) \ . \tag{31}$$

Here $G_1(x,\omega)=G(x,\omega)/N$ is the single particle gain, N is the particle number in the beam and $\varepsilon(\omega_n)$ in the denominator takes into account particle screening[3] (see Eq. (12)).

Similar to the drag force the particle diffusion can be considered as a single particle effect. First, we relate $D(x)$ to the growth rate of momentum spread for initial point-like momentum distribution, $f(x) = \delta(x-x_0)$. Multiplying Eq. (30) by $(x-x_0)^2$ and integrating one obtains:

$$\frac{d}{dt}\overline{(x-x_0)^2} = \int (x-x_0)^2 \frac{\partial f}{\partial t}dx = -\int (x-x_0)^2 \frac{\partial}{\partial x}\left(F(x)f\right)dx + \frac{1}{2}\int (x-x_0)^2 \frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right)$$
$$= 2\int (x-x_0)F(x)f(x)dx + \int f(x)\left(D(x) + (x-x_0)\frac{dD}{dx}\right)dx \xrightarrow{f(x)=\delta(x-x_0)} D(x) \quad . \tag{32}$$

Second, we consider the single particle diffusion due to kicker noise. Let the particle energy at turn n be equal to $\delta E_n = eU(nT)$, where T is the particle revolution period, and $U(t)$ is the kicker voltage. Then energy spread after N_r turns is:

$$\overline{(\Delta E)^2} = \overline{\left(e\sum_{n=0}^{N_r-1} U(nT)\right)^2} = e^2\sum_{n=0}^{N_r-1}\sum_{m=0}^{N_r-1}\overline{U(nT)U(mT)} = e^2\sum_{n=0}^{N_r-1}\sum_{m=0}^{N_r-1} K_A\left((n-m)T\right)$$
$$= e^2\sum_{n=0}^{N_r-1}\sum_{m=0}^{N_r-1}\int_{-\infty}^{\infty} P_A(\omega)e^{i\omega T(n-m)}d\omega = e^2\int_{-\infty}^{\infty} P_A(\omega)\left|\frac{e^{i\omega T N_r}-1}{e^{i\omega T}-1}\right|^2 d\omega \xrightarrow{N_r\gg 1} \frac{2\pi e^2 N_r}{T}\sum_{n=-\infty}^{\infty} P_A\left(\frac{2\pi}{T}n\right) \quad . \tag{33}$$

where $K_A(t)$ and $P_A(t)$ are the correlation function and the spectral density of kicker accelerating voltage so that

$$K_A(t) = \int_{-\infty}^{\infty} P_A(\omega) e^{i\omega t} d\omega \quad . \tag{34}$$

Comparing Eqs. (32) and (33) and taking into account dependence of revolution period T on the momentum and the relationship between the relative energy and momentum deviations one obtains:

$$D(x) = \frac{2\pi e^2}{T_0^{\ 2}\left(\gamma\beta^2 mc^2\right)^2} \sum_{n=-\infty}^{\infty} \left| \frac{Z_k(\omega_n)}{Z_{ampl}} \right|^2 P_U(\omega_n) \quad , \tag{35}$$

where we neglected difference between T and T_0 in the denominator, and took into account the relationship between the kicker accelerating voltage and the voltage of power amplifier, $P_A(\omega) = \left| Z_k(\omega) / Z_{ampl} \right|^2 P_U(\omega)$.

The spectral density of the kicker noise consists of two contributions. The first one is related to the noise of electronics at the exit of power amplifier, P_{Unoise}, and the second one is related to the particle noise. The beam current shot noise for non-interacting particles is equal to:

$$P_I(\omega) = \frac{e^2 N}{2\pi T_0} \sum_{k=-\infty}^{\infty} \frac{1}{|k\eta|} f\left(\frac{k\omega_0 - \omega}{\eta k \omega_0} \right) \quad . \tag{36}$$

Taking into account the cooling system amplification and the Schottky noise suppression by the particle interaction we obtain an expression for the diffusion coefficient:

$$\begin{aligned} D(x) = \frac{2\pi e^2}{T_0^{\ 2}\left(\gamma\beta^2 mc^2\right)^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left|\varepsilon(\omega_n)\right|^2} & \left[\left| \frac{Z_k(\omega_n)}{Z_{ampl}} \right|^2 P_{Unoise}(\omega_n) \right. \\ & \left. + \left| \frac{Z_p(x,\omega_n) Z_k(\omega_n)}{Z_{ampl}} K(\omega_n)\left(1 - A(\omega_n) e^{-i\omega_n T_0}\right) \right|^2 \frac{e^2 N}{2\pi T_0} \sum_{k=-\infty}^{\infty} \frac{1}{|k\eta|} f\left(\frac{k - (1-\eta x) n}{\eta k} \right) \right] \quad . \end{aligned} \tag{37}$$

After simplification it yields:

$$\begin{aligned} D(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\left|\varepsilon(\omega_n)^2\right|} & \left[\frac{2\pi e^2 P_{Unoise}(\omega_n)}{T_0^{\ 2}\left(\gamma\beta^2 mc^2\right)^2} \left| \frac{Z_k(\omega_n)}{Z_{ampl}} \right|^2 + \right. \\ & \left. \frac{N}{T_0} \left| G_1(x,\omega_n)\left(1 - A(\omega_n) e^{-i\omega_n T_0}\right) \right|^2 \sum_{k=-\infty}^{\infty} \frac{1}{|k\eta|} f\left(\frac{k - (1-\eta x) n}{\eta k} \right) \right] \quad . \end{aligned} \tag{38}$$

where the same as for in Eq.(31) $\omega_n \equiv \omega_n(x) = n\omega_0(1 - \eta x)$.

4. Fokker-Planck Equation for Transverse Cooling

Natural variables for transverse cooling description are the action-phase variables (I, ψ). We determine the action so that

$$I = \frac{1}{2}\left(\beta_y \theta^2 + 2\alpha_y y\theta + \frac{1+\alpha_y^{\ 2}}{\beta_y} y^2 \right) \quad , \tag{39}$$

where β_y and α_y are the beta- and alpha-functions of the ring. We assume that there is no x-y coupling in the lattice, and the cooling is linear in betatron amplitude, which, in practical terms, means that the electronics is not saturated and the pickup has linear response across its aperture. That yields that the beam distribution function can be described by the following equation:

$$\frac{\partial f_\perp}{\partial t}+\lambda_\perp(x)\frac{\partial}{\partial I}\left(If_\perp\right)=D_\perp(x)\frac{\partial}{\partial I}\left(I\frac{\partial f_\perp}{\partial I}\right) \quad . \tag{40}$$

Here $f_\perp \equiv f_\perp(x,I,t)$ is the distribution function normalized so that $\int f_\perp(x,I,t)dI = f_0(x)$ and the same as above $\int f_0(x)dx=1$, $\lambda_\perp(x)$ is the cooling decrement, and $D_\perp(x)$ is the diffusion coefficient. $\lambda_\perp(x)$ and $D_\perp(x)$ do not depend on I because of system linearity on the transverse coordinate y.

Similar to the longitudinal cooling the transverse cooling is created by the particle self-interaction and therefore is not directly affected by the band overlap but is still affected by the screening. To find the cooling decrement, first, we consider a single particle damping. Introducing the ring transfer matrix, $\mathbf{M} = \mathbf{M}_1\,\mathbf{M}_2$, and the partial kicker-to-pickup and pickup-to-kicker transfer matrices, $\mathbf{M}_1$, $\mathbf{M}_2$, and using the normalized transverse coordinates introduced in Section 2 we can write the total ring (pickup-to-pickup) transfer matrix in the following form:

$$\mathbf{M}_{tot}=\mathbf{M}_1\left(\mathbf{M}_2+\mathbf{G}\right)=\mathbf{M}+\mathbf{M}_1\mathbf{G}=\begin{bmatrix}\cos(2\pi\nu)+G_{\perp 1}\sin(2\pi\nu_1) & \sin(2\pi\nu)\\ -\sin(2\pi\nu)+G_{\perp 1}\cos(2\pi\nu_1) & \cos(2\pi\nu)\end{bmatrix} \quad . \tag{41}$$

Here we also took into account that the angle change in the kicker is proportional to the particle displacement in the pickup:

$$\begin{bmatrix}\delta\tilde{x}\\ \delta\tilde{\theta}\end{bmatrix}_{kicker}=\begin{bmatrix}0\\ G_{\perp 1}x_{pickup}\end{bmatrix}=\begin{bmatrix}0 & 0\\ G_{\perp 1} & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{\theta}\end{bmatrix}_{pickup}\equiv\mathbf{G}\begin{bmatrix}\tilde{x}\\ \tilde{\theta}\end{bmatrix}_{pickup} \quad , \tag{42}$$

and $G_{\perp 1}$ is the system gain. Damping of the cooling system is determined by the eigen-values of the total ring transfer matrix, $\mathbf{M}_{tot}$. The eigen-values are:

$$\Lambda_{1,2}=\cos(2\pi\nu)+G_{\perp 1}\frac{\sin(2\pi\nu_1)}{2}\pm\sqrt{\left(\cos(2\pi\nu)+G_{\perp 1}\frac{\sin(2\pi\nu_1)}{2}\right)^2+\left(1-G_{\perp 1}\sin(2\pi\nu_2)\right)} \quad . \tag{43}$$

Taking into account that the single particle gain is small, $G_{\perp 1}<<1$, and leaving only linear term in Taylor expansion of Eq. (43) one obtains:

$$\Lambda_{1,2}\approx e^{\pm 2\pi i\nu}\left(1+\frac{G_{\perp 1}}{2i}e^{-2\pi i\nu_2}\right) \quad , \quad G_{\perp 1}<<1 \quad . \tag{44}$$

That yields the damping decrements of both transverse modes are expressed by the same equation:

$$\lambda_{1,2}=-2\ln\left|\Lambda_{1,2}\right|\approx\mathrm{Re}\left(iG_{\perp 1}e^{-2\pi i\nu_2}\right) \quad , \quad G_{\perp 1}<<1 \quad . \tag{45}$$

Here factor of 2 takes into account difference between damping decrements for the amplitude and the action. Expending particle signal in Fourier harmonics and summing their effect on the particle motion we finally obtain[1]:

$$\lambda_\perp(x)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\mathrm{Re}\left(i\frac{G_{\perp 1}(\omega_n)}{\varepsilon_\perp(\omega_n)}e^{i\omega_n T_2\eta_2 x-2\pi i\nu_2}\right) \quad , \quad \omega_n=\omega_0\left(n(1-\eta x)-(\nu+\xi x)\right) \quad , \tag{46}$$

where $G_{\perp 1}(\omega)=G_{\perp}(\omega)/N$ is the single particle gain, $\varepsilon_{\perp}(\omega)$ takes into account screening of the particle field, and the term $e^{i\omega_n T_2\eta_2 x}$ takes into account changes of the particle arrival time to the kicker.

The diffusion coefficient is obtained similar to Eqs. (32) – (35). That yields:

$$D_{\perp}(x)=\frac{\pi\beta_k}{2T_0^{\;2}}\sum_{n=-\infty}^{\infty}P_{\theta}(\omega_n) \quad , \tag{47}$$

where $P_{\theta}(\omega)$ is the spectral density of the angle kicks produced by the kicker. $P_{\theta}(\omega)$ consists of two contributions: the spectral density of amplifier noise, $P_{\perp Ua}(\omega)$, and the amplified shot noise of the beam. The shot noise of the beam at the pickup is

$$P_{\perp Up}(\omega)=\frac{e^2\left|Z_{p\perp}(\omega)\right|^2\overline{y^2}}{T_0^{\;2}}N\sum_{m=-\infty}^{\infty}\frac{1}{\omega_0\left|\xi+\eta m\right|}f\left(\frac{\omega_0(m-\nu)-\omega}{\omega_0(\xi+\eta m)}\right) , \tag{48}$$

Substituting Eq. (48) into Eq. (47), taking into account particle screening and using definition of the single particle gain we obtain:

$$D_{\perp}(x)=\sum_{n=-\infty}^{\infty}\frac{1}{\left|\varepsilon_{\perp}(\omega_n)\right|^2}\left(\frac{\pi\beta_k}{2T_0^{\;2}}\left(\frac{e\left|Z_{k\perp}(\omega_n)\right|}{mc^2\beta^2\gamma Z_{ampl}}\right)^2P_{\perp U}(\omega_n)+\left|G_{\perp 1}(\omega_n)\right|^2\frac{\overline{I(x)}\,N}{2T_0}\sum_{m=-\infty}^{\infty}\frac{f\left(\dfrac{m-n+(\xi+\eta m)x}{\xi+\eta m}\right)}{\left|\xi+\eta m\right|}\right), \tag{49}$$

where $\overline{I(x)}=\int f_{\perp}(x,I,t)IdI$ is the average action for given momentum deviation x.

For the case of non-overlapping bands the above equation can be simplified[1]:

$$D_{\perp}(x)=\sum_{n=-\infty}^{\infty}\frac{1}{\left|\varepsilon_{\perp}(\omega_n)\right|^2}\left(\frac{\pi\beta_k}{2T_0^{\;2}}\left(\frac{e\left|Z_{k\perp}(\omega_n)\right|}{mc^2\beta^2\gamma Z_{ampl}}\right)^2P_{\perp U}(\omega_n)+\left|G_{\perp 1}(\omega_n)\right|^2\frac{\langle I(x)\rangle N}{2T_0}\frac{f(x)}{\left|\xi+\eta n\right|}\right) \quad . \tag{50}$$

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